Supplemental material to the article

Nonlinear generation of vorticity in thin smectic films

1. Bending and Longitudinal sounds. Based on Eqs. (1), (3) from the main text, we can obtain equations describing bending and longitudinal sounds. We should find the solution of exact equations (1), (2) in the linear approximation. We expand two-dimensional density of the film in a series of small parameter $|\nabla h| \ll 1$, i.e. $\rho = \rho_0 + \rho^{(1)} + \cdots$. Then in the linear approximation the Eq. (2) gives

$$\rho_0 \partial_t v_\alpha = -(B/\rho_0)\partial_\alpha \rho^{(1)} + \zeta \partial_\alpha \partial_\beta v_\beta + \eta \nabla^2 v_\alpha, \tag{S1}$$

$$\rho_0 \partial_t v_z = \sigma_0 \nabla^2 h, \tag{S2}$$

where we have substituted

$$\sigma = \sigma_0 - B\left(\frac{\rho}{\rho_0\sqrt{g}} - 1\right). \tag{S3}$$

Here B is the film compressibility module, ρ_0 is the equilibrium mass density of the film and σ_0 is the equilibrium surface tension. Next, using the linearized boundary conditions (1), we find

$$\rho_0 \partial_t^2 \rho^{(1)} = B\nabla^2 \rho^{(1)} + (\zeta + \eta)\partial_t \nabla^2 \rho^{(1)}, \tag{S4}$$

$$\rho_0 \partial_t^2 h = \sigma_0 \nabla^2 h. \tag{S5}$$

The first equation corresponds to the longitudinal sound, that is a motion in the film plane, and it obeys the dispersion law $\omega = \pm k\sqrt{B/\rho_0} - ik^2(\zeta+\eta)/2\rho_0$. The second equation describes the bending sound, characterized by the dispersion law $\omega = \pm k\sqrt{\sigma_0/\rho_0}$ and discussed in the paper. In the linear approximation these two modes are independent from each other. We also assume that the longitudinal sound does not excited by the pumping force directly, i.e. we set $\rho^{(1)} = 0$.

2. Vorticity in the film surrounded by vacuum. Now we consider the vertical component of the vorticity $\varpi_z = \epsilon_{\beta\gamma}\partial_\beta v_\gamma$. In the linear approximation ϖ_z is zero, since the motion of liquid in the film plane is not generated by the bending mode. Therefore to find ϖ_z we should go beyond the linear approximation. We take into account the main nonlinear contribution to ϖ_z, which is of the second order in the film elevation. Using the Eq. (2) and the equality $\rho^{(1)} = 0$, we obtain

$$(\rho_0/\eta)\partial_t \varpi_z - \nabla^2 \varpi_z = -(\sigma_0/\eta)\epsilon_{\beta\gamma}\partial_\gamma h \partial_\beta \nabla^2 h + \epsilon_{\beta\gamma}\partial_\beta \partial_\alpha \partial_t(\partial_\alpha h \partial_\gamma h), \tag{S6}$$

where we have already substituted v_z by $\partial_t h$ in nonlinear terms based on the linearized Eq. (1). Further we assume that the external pumping is monochromatic and we consider only the steady contribution to the excited vorticity ϖ_z. After averaging over time we pass to the equation

$$\nabla^2 \varpi_z = (\sigma_0/\eta)\epsilon_{\beta\gamma}\langle \partial_\gamma h \partial_\beta \nabla^2 h\rangle, \tag{S7}$$

which is written in the main text. Note that the right-hand-side is zero in the dissipationless case. Now we take into account the attenuation of the bending mode. Let us consider the case, where the film displacement is approximately a superposition of two standing waves

$$h = H_1 \sin(k_x x)\sin(k_y y)\cos(\omega t) + H_2 \sin(q_x x)\sin(q_y y)\cos(\omega t + \phi), \quad k_x^2 + k_y^2 = q_x^2 + q_y^2 = |k|^2. \tag{S8}$$

To take into account the attenuation explicitly, one has to expand the expression over the plane waves

$$h = -\frac{H_1}{4}\cos(k_x x + k_y y - \omega t)e^{-\beta(k_x x + k_y y)} - \frac{H_1}{4}\cos(-k_x x - k_y y - \omega t)e^{\beta(k_x x + k_y y)} +$$

$$+\frac{H_1}{4}\cos(k_x x-k_y y-\omega t)e^{-\beta(k_x x-k_y y)}+\frac{H_1}{4}\cos(-k_x x+k_y y-\omega t)e^{\beta(k_x x-k_y y)}-$$
$$-\frac{H_2}{4}\cos(q_x x+q_y y-\omega t-\phi)e^{-\beta(q_x x+q_y y)}-\frac{H_2}{4}\cos(-q_x x-q_y y-\omega t-\phi)e^{\beta(q_x x+q_y y)}+$$
$$+\frac{H_2}{4}\cos(q_x x-q_y y-\omega t-\phi)e^{-\beta(q_x x-q_y y)}+\frac{H_2}{4}\cos(-q_x x+q_y y-\omega t-\phi)e^{\beta(q_x x-q_y y)}, \quad (S9)$$

where the damping constant $\beta \ll 1$ should be obtained from the modified dispersion law. The different mechanisms contributed to the constant β have been discussed in the main text. Next, we substitute the Eq. (S9) into the Eq. (S7) and obtain

$$\varpi_z=\frac{2\beta\sigma_0}{\eta}H_1H_2\frac{|k|^2}{\hat{k}^2}\sin\phi\Big[k_yq_x\sin(k_x x)\sin(q_y y)\cos(q_x x)\cos(k_y y)-k_xq_y\cos(k_x x)\cos(q_y y)\sin(q_x x)\sin(k_y y)\Big]. \quad (S10)$$

Qualitatively, the spatial structure is similar to the Fig. 1, presented in the main text. The vorticity amplitude provides information about the attenuation constant β of the bending sound. Note that $\beta=\alpha/\omega$, where the constant α is defined in the main text, see Eq. (4).

3. Bending mode for the film surrounded by air. The linearized Navier–Stokes equation takes a form $\partial_t\mathbf{v}=-\nabla P/\rho_a+\nu_a\nabla^2\mathbf{v}$ and it should be supplemented by the incompressibility condition $\operatorname{div}\mathbf{v}=0$. Taking the divergence of the equation we find that the pressure P should be a solution of the Laplace equation. Thus,

$$P=P_2e^{i\mathbf{k}r-i\omega t}e^{-|k|z},\ z>0\quad\text{and}\quad P=P_1e^{i\mathbf{k}r-i\omega t}e^{|k|z},\ z<0, \quad (S11)$$

and then the linearized Navier–Stokes equation is

$$\begin{cases}(\partial_t+\nu_ak^2-\nu_a\partial_z^2)v_\alpha=-ik_\alpha P_2e^{-|k|z}/\rho_a,\\(\partial_t+\nu_ak^2-\nu_a\partial_z^2)v_z=|k|P_2e^{-|k|z}/\rho_a,\end{cases} z>0\quad\text{and}\quad\begin{cases}(\partial_t+\nu_ak^2-\nu_a\partial_z^2)v_\alpha=-ik_\alpha P_1e^{|k|z}/\rho_a,\\(\partial_t+\nu_ak^2-\nu_a\partial_z^2)v_z=-|k|P_1e^{|k|z}/\rho_a,\end{cases} z<0, \quad (S12)$$

The system has a solution, which is a sum of forced (potential) and eigen (solenoidal) terms

$$\begin{cases}v_\alpha=\frac{k_\alpha P_2}{\rho_a\omega}e^{-|k|z}+\kappa A_\alpha e^{-\kappa z},\\ v_z=\frac{i|k|P_2}{\rho_a\omega}e^{-|k|z}+ik_\alpha A_\alpha e^{-\kappa z},\end{cases} z>0\quad\text{and}\quad\begin{cases}v_\alpha=\frac{k_\alpha P_1}{\rho_a\omega}e^{|k|z}+\kappa B_\alpha e^{\kappa z},\\ v_z=\frac{-i|k|P_1}{\rho_a\omega}e^{|k|z}-ik_\alpha B_\alpha e^{\kappa z},\end{cases} z<0, \quad (S13)$$

where we have used the incompressibility condition $ik_\alpha v_\alpha+\partial_zv_z=0$ and we have also introduced $\kappa^2=k^2-i\omega/\nu_a$. To find the values of constants A and B we should consider the motion in the film plane. From the mass conservation law (1), $\partial_\alpha v_\alpha=0$ at $z=0$, we obtain

$$A_\alpha=-\frac{k_\alpha P_2}{\rho_a\omega}\frac{1}{\kappa},\quad B_\alpha=-\frac{k_\alpha P_1}{\rho_a\omega}\frac{1}{\kappa}. \quad (S14)$$

The continuity of the velocity component v_z at the interface $z=0$ leads to the condition $P_1=-P_2=P_0$. Finally, we find

$$v_\alpha=\mp\frac{k_\alpha P_0}{\rho_a\omega}\left(e^{\mp|k|z}-e^{\mp\kappa z}\right),\quad v_z=\frac{-i|k|P_0}{\rho_a\omega}\left(e^{\mp|k|z}-\frac{|k|}{\kappa}e^{\mp\kappa z}\right), \quad (S15)$$

where upper (lower) sign corresponds to the region $z>0$ ($z<0$). The relation between the pressure P_0 and the film elevation h can be obtained from the kinematic boundary condition (1) $\partial_th=v_z$ posed at $z=0$

$$P_0=-\nu_a\rho_a\frac{\kappa(\kappa+|k|)}{|k|}\partial_th. \quad (S16)$$

Substituting the Eq. (S16) into the Eq. (S15), we obtain the formula for the velocity field

$$v_\alpha = \mp\nu_a \frac{\hat{\kappa}(\hat{\kappa}+\hat{k})}{\hat{k}} \left(e^{\mp\hat{k}z} - e^{\mp\hat{\kappa}z}\right)\partial_\alpha h, \quad v_z = \nu_a(\hat{\kappa}+\hat{k})\left(\hat{\kappa}e^{\mp\hat{k}z} - \hat{k}e^{\mp\hat{\kappa}z}\right)h, \tag{S17}$$

which is written in the main text. The dispersion law for the bending mode can be found from the Eq. (9) for the momentum density j_z. In the linear approximation it reads

$$\rho_0\partial_t^2 h = \sigma_0\nabla^2 h + 2P_0, \tag{S18}$$

and then, substituting the Eq. (S16) into the Eq. (S18), we obtain

$$\omega^2 = \omega_0^2\left(1 - \frac{i\gamma}{\sqrt{2}}\Theta\right),\ \omega_0^2 = \frac{\sigma_0|k|^2}{\rho_0 + 2\rho_a/|k|},\ \Theta = \frac{2\rho_a/|k|}{\rho_0+2\rho_a/|k|}, \tag{S19}$$

where $\gamma = \sqrt{\nu_a k^2/\omega_0} \ll 1$.

4. Details of derivation the expression for vorticity. As it was explained in the main text, in order to obtain the steady vertical vorticity one should solve the equation

$$(\partial_z^2 - \hat{k}^2)\varpi_z = -f, \quad f = \nu_a^{-1}\langle\varpi_\alpha\partial_\alpha v_z\rangle, \tag{S20}$$

with the boundary condition $\langle(\partial_z\varpi_z)^{\rm II} - (\partial_z\varpi_z)^{\rm I}\rangle = 0$ posed at $z=0$. The solution of the equation is $\varpi_z = e^{\hat{k}z}A(z) + e^{-\hat{k}z}B(z)$, where $\partial_z A = -\hat{k}^{-1}e^{-\hat{k}z}(f/2)$, $\partial_z B = \hat{k}^{-1}e^{\hat{k}z}(f/2)$. Up to the first two orders in the parameter γ, we obtain

$$\varpi_\alpha = \epsilon_{\alpha\beta}\frac{\hat{\kappa}+\hat{k}}{\hat{k}}e^{\mp\hat{\kappa}z}\partial_\beta\partial_t h, \quad f = \epsilon_{\alpha\beta}\left\langle\left[\frac{\hat{\kappa}+\hat{k}}{\hat{k}}e^{\mp\hat{\kappa}z}\partial_\beta\partial_t h\right]\left[(\hat{\kappa}+\hat{k})(\hat{\kappa}e^{\mp\hat{k}z} - \hat{k}e^{\mp\hat{\kappa}z})\partial_\alpha h\right]\right\rangle, \tag{S21}$$

and thus

$$\varpi_z = e^{\mp\hat{k}z}C + \left\langle \frac{\epsilon_{\alpha\beta}\hat{\kappa}_1\hat{\kappa}_2\hat{k}_2}{(\hat{\kappa}_1+\hat{\kappa}_2)^2\hat{k}_1}e^{\mp(\hat{\kappa}_1+\hat{\kappa}_2)z}(\partial_\beta\partial_t h)(\partial_\alpha h) - \right.$$
$$\left. -\frac{\epsilon_{\alpha\beta}\hat{\kappa}_2^2}{\hat{\kappa}_1\hat{k}_1}\left(1 + \frac{\hat{k}_2}{\hat{\kappa}_2} + \frac{\hat{k}_1}{\hat{\kappa}_1} - \frac{2\hat{k}_2}{\hat{\kappa}_1}\right)e^{\mp(\hat{\kappa}_1+\hat{k}_2)z}(\partial_\beta\partial_t h)(\partial_\alpha h)\right\rangle, \tag{S22}$$

where we have taken into account the continuity of ϖ_z at the interface $z=0$. Hereinafter the operator with i subscript acts only on the i parenthesis containing the film elevation h. The constant C is defined from the boundary condition. With the same accuracy, we find

$$C = (\nu_a\hat{k})^{-1}\epsilon_{\alpha\beta}\left\langle\left(\hat{k}^{-1}\partial_\beta\partial_t h\right)\partial_\alpha\partial_t h\right\rangle. \tag{S23}$$

Substituting the Eq. (S23) into the Eq. (S22) we obtain the answer which is written in the main text

$$\varpi_z = \epsilon_{\alpha\beta}\left\langle\left(\frac{\hat{\kappa}}{\hat{k}}e^{\mp\hat{\kappa}z}\partial_\alpha h\right)e^{\mp\hat{k}z}\partial_\beta\partial_t h\right\rangle + (\nu_a\hat{k})^{-1}e^{\mp\hat{k}z}\epsilon_{\alpha\beta}\left\langle(\hat{k}^{-1}\partial_\beta\partial_t h)\partial_\alpha\partial_t h\right\rangle. \tag{S24}$$