

VORTEX DYNAMICS IN CLASSICAL NON-ABELIAN SPIN MODELS

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We discuss the abelian vortex dynamics in the abelian projection approach to non-abelian spin models. We show numerically that in the three-dimensional $SU(2)$ spin model in the Maximal Abelian projection the abelian off-diagonal vortices are not responsible for the phase transition contrary to the diagonal vortices. A generalization of the abelian projection approach to $SU(N)$ spin models is briefly discussed.

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One of the important phenomena in spin models is the mass gap generation. This phenomena have intensively been studied for the abelian spin models in various space dimensions. Classical papers by Berezinskii, Kosterlitz and Thouless [1] show that in 2D XY model the mass gap generation is due to the condensation of topological excitations called vortices. In a massive phase the vortices are condensed and this leads to the exponential falloff of the two-point correlation function while at the weak coupling the vortices form a dilute gas with the logarithmic long-range interaction (see [2] for a rigorous proof). In the three-dimensional XY model the condensation of vortices also leads to the mass gap generation [3, 4]. In this case the situation is somewhat simpler since one has a spontaneous breaking of the global $U(1)$ symmetry at the weak coupling phase and vortices interact via the short-range Coulomb potential above the critical point.

The nature of mass gap in non-abelian spin systems is still an open question despite numerous attempts to attack this problem [5]. The non-abelian models, similarly to their gauge partners, possess a number of excitations (instantons, merons, thin and thick vortices) and this is a delicate problem to estimate their contribution to the mass gap reliably. Nevertheless, one can handle the problem, at least partially, using the gauge fixing procedure and calculating the contribution to the mass gap from configurations which the gauge fixing leaves untouched. This idea came obviously up from lattice gauge models where it was successfully applied for studying the confinement mechanism. We put forward recently such an approach in Ref. [6] making use the abelian projection method [7]. After abelian projection the non-abelian spin system possesses abelian symmetries and abelian topological excitations (vortices). It is important to stress at this point that, precisely like in gauge models, the abelian projection does not mean the restriction of the configuration space of the original non-abelian model. The original non-abelian global symmetry is broken up to its maximal abelian subgroup by appropriate gauge fixing procedure. In general, global gauge fixing procedure is a powerful tool for investigating nonabelian models in different regimes. Also it was understood pretty long ago that this

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is a necessary condition for construction of the correct perturbation theory of non-abelian models in the weak coupling region [8]. We would like to argue that this method is also very fruitful in studying the nonperturbative phenomena like the mass gap generation.

Following this avenue, we have found the abelian dominance [9] in the Maximal Abelian (MaA) projection²⁾ of the $SU(2)$ spin model: the full mass gap (i.e., calculated from full correlation function) in classical non-abelian $SU(2)$ spin model is dominated by the abelian mass gap calculated from the projected correlation function [6]. We have concluded that the abelian degrees of freedom seem to play an important role in the mass gap generation phenomena.

The next important question is what is the role of abelian topological excitations in the phenomena of the abelian dominance. The 3D $SU(2)$ spin model possesses two types (“diagonal” and “off-diagonal”) of the abelian vortices. The diagonal vortices were shown to be condensed in the massive phase and they form the dilute gas in the massless phase [6]. This picture is very similar to the vortex dynamics in the abelian XY model. In this paper we study the behavior of the off-diagonal vortices across the phase transition. The generalization of the abelian projection approach to the $SU(N)$ spin models is discussed at the end of the paper.

We study the three-dimensional $SU(2)$ spin model with the following action

$$S = -\frac{\beta}{2} \sum_{\mathbf{x}} \sum_{\mu=1}^3 \text{Tr} U_{\mathbf{x}} U_{\mathbf{x}+\mu}^+, \quad (1)$$

where $U_{\mathbf{x}}$ are the spin fields taking values in the $SU(2)$ group and β is the coupling constant. Action (1) is invariant under $SU_L(2) \times SU_R(2)$ global transformations, $U_{\mathbf{x}} \rightarrow U_{\mathbf{x}}^{(\Omega)} = \Omega_L^+ U_{\mathbf{x}} \Omega_R$, where $\Omega_{L,R}$ are the $SU(2)$ matrices.

The MaA projection for the $SU(2)$ spin theory is defined by the following maximization condition [6]:

$$\max_{\{\Omega\}} R[U^{(\Omega)}], \quad R[U] = \sum_{\mathbf{x}} \text{Tr} \left(U_{\mathbf{x}} \sigma^3 U_{\mathbf{x}}^+ \sigma^3 \right). \quad (2)$$

The functional $R[U]$ is invariant under $U_L(1) \times U_R(1)$ global transformations, $U_{\mathbf{x}} \rightarrow U_{\mathbf{x}}^{(\tilde{\Omega})} = \tilde{\Omega}_L^+ U_{\mathbf{x}} \tilde{\Omega}_R$, where $\tilde{\Omega}_{L,R} = e^{i\sigma^3 \omega_{L,R}}$ and $\omega_{L,R} \in [0, 2\pi)$. Due to the invariance of the functional R under the abelian gauge transformations, the condition (1) fixes the $SU_L(2) \times SU_R(2)$ global symmetry group up to $U_L(1) \times U_R(1) \sim O_L(2) \times O_R(2)$ global group.

In order to fix the MaA gauge in the path integral approach we substitute the Faddeev – Popov (FP) unity

$$1 = \Delta_{FP}[U; \lambda] \int D\Omega \exp\{\lambda R[U]\}, \quad D\Omega = D\Omega_L D\Omega_R \quad (3)$$

into the partition function of $SU(2)$ spin model. Here Δ_{FP} is the FP determinant and the limit $\lambda \rightarrow +\infty$ is assumed. The functional $R[U]$ is defined in (1). Shifting the fields $U_{\mathbf{x}} \rightarrow U_{\mathbf{x}}^{\Omega^+}$ and using the invariance of the FP determinant $\Delta_{FP}[U; \lambda]$, the action $S[U]$ and the Haar measure DU , we get the product of the gauge orbit volume $\int D\Omega$ and the

²⁾ The MaA projection was first used for study of the lattice non-abelian gauge theories in Ref. [10].

partition function with the fixed gauge:

$$Z_{MaA} = \int DU \exp\{-S[U] + \lambda R[U]\} \Delta_{FP}[U; \lambda]. \quad (4)$$

The limit $\lambda \rightarrow \infty$ should guarantee that all the saddle points of invariant integral in (4) lie in the abelian subgroup.

Let us parametrize the $SU(2)$ spin field U in the standard way: $U_x^{11} = \cos \varphi_x e^{i\theta_x}$; $U_x^{12} = \sin \varphi_x e^{i\chi_x}$; $U_x^{22} = U_x^{11,*}$; $U_x^{21} = -U_x^{12,*}$; $0 \leq \varphi \leq \pi/2$, $0 < \theta, \chi \leq 2\pi$. Under the $U_L(1) \times U_R(1)$ transformations defined above, components of the field U transform as

$$\theta_x \rightarrow \theta'_x = \theta_x + \omega_d \pmod{2\pi}, \quad \chi_x \rightarrow \chi'_x = \chi_x + \omega_o \pmod{2\pi}, \quad (5)$$

where $\omega_{d,o} = -\omega_L \mp \omega_R$. It is convenient to decompose the residual symmetry group, $O_L(2) \times O_R(2) \sim O_d(2) \times O_o(2)$. The diagonal (off-diagonal) component θ (χ) of the $SU(2)$ spin U transforms as a spin variable with respect to the $O_d(2)$ ($O_o(2)$) symmetry group.

It is useful to consider the one-link spin action S_l in terms of the angles φ , θ , χ :

$$S_{l_{x,\mu}} = -\beta \left[\cos \varphi_x \cos \varphi_{x+\mu} \cos(\theta_x - \theta_{x+\mu}) + \sin \varphi_x \sin \varphi_{x+\mu} \cos(\chi_x - \chi_{x+\mu}) \right]. \quad (6)$$

The action consists of two parts which correspond to the self-interaction of the spins θ and χ , respectively. The $SU(2)$ component φ does not behave as a spin field and its role is to provide the interaction between the θ and χ spins. Thus, the $SU(2)$ spin model in the abelian projection reduces to two interacting copies of the XY model with the fluctuating couplings due to the dynamics of the field φ .

The phases $\theta = \arg U^{11}$ and $\chi = \arg U^{12}$ behave as the $O(2)$ spins under the abelian gauge transformation [6]. In the abelian projection the $SU(2)$ spin model possesses two types of the abelian vortices due to the compactness of the residual abelian group. These vortices correspond to the diagonal (θ) and to the off-diagonal (χ) abelian spins ("diagonal" and "off-diagonal" vortices, respectively).

We expect that in the MaA projection the diagonal vortices may be more dynamically important than the off-diagonal vortices. The reason for this expectation is simple. In our representation for $SU(2)$ spin field the maximizing functional (1) has the form: $R = 4 \sum_x \cos^2 \varphi_x + \text{const}$. Therefore, in the MaA projection the effective coupling constant in front of the action for the θ spins is maximized while the effective self-coupling for χ spins is minimized. Thus we may expect that the diagonal vortices may be more relevant to the dynamics of the system than the off-diagonal vortices.

The behavior of the diagonal vortices was studied numerically in Ref. [6]. The diagonal abelian vortices are condensed in the massive phase and they form the dilute gas of the vortex anti-vortex pairs in the Coulomb phase. This behavior is very similar to the behavior of the abelian vortices in 3D XY model [3, 4], where the vortices are known to be responsible for the mass gap generation. This similarity allows to conclude that the diagonal abelian vortices in the MaA projection of $SU(2)$ spin model are relevant degrees of freedom for the mass gap generation [6].

Now we proceed to study the condensation properties of the off-diagonal vortices in the MaA projection across the phase transition. First, we construct the off-diagonal vortex trajectory $*j_o$ from the spin variables χ using the standard formula [11, 12]: $*j_o =$

$= (2\pi)^{-1} d[d\chi]$, where the square brackets stand for “modulo 2π ” and the operator “ d ” is the lattice derivative. In the three dimensions the vortex trajectories are loops which are closed due to the property $d^2 = 0$.

The condensation of the vortex trajectories can be studied by measuring of the so-called percolation probability [4] C :

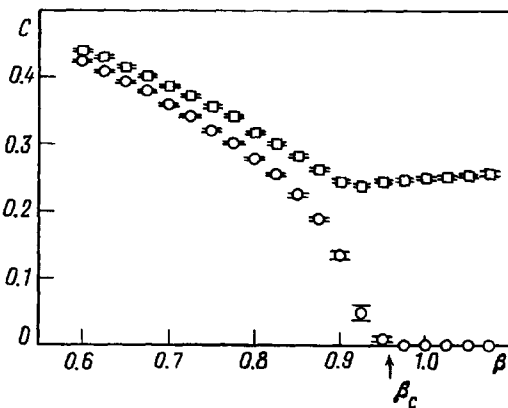
$$C = \lim_{r \rightarrow \infty} \left(\sum_{x,y,i} \delta_{x \in j_i} \delta_{y \in j_i} \delta(|x - y| - r) \right) \left(\sum_{x,y} \delta(|x - y| - r) \right)^{-1}, \quad (7)$$

where the summation is over all the vortex trajectories j_i , and over all the points x, y of the lattice.

If there is a non-zero probability for two infinitely separated points to be connected by a vortex trajectory then the quantity C is not zero. This means that the entropy of the vortex trajectories dominates over the energy of vortices, which means in turn that the vortex condensate exists in the vacuum. If the vortex trajectories form an ensemble of small loops (the gas of vortex-anti-vortex pairs) then the quantity C is obviously zero.

We studied the quantity C by numerical methods on the 16^3 lattice with periodic boundary conditions. In our numerical simulations the Wolff cluster algorithm [13] has been used. In order to thermalize the spin fields at each value of the coupling constant β we performed a number of thermalization sweeps, which is much greater than measured auto-correlation time.

We show the percolation probability C for both diagonal (taken from Ref. [6]) and off-diagonal vortex trajectories in Figure. One can see that the percolation probability of the off-diagonal vortex trajectories does not vanish in the massless phase contrary to that of diagonal vortices. This allows us to conclude that the off-diagonal vortices do not play a significant role in the mass gap generation³⁾ and that the only diagonal sector of the $SU(2)$ spin model in the MaA projection may be responsible for the mass gap.



The percolation probability C for the diagonal (circles) and anti-diagonal (squares) abelian vortex trajectories vs. β on the 16^3 lattice. The data for diagonal vortex trajectories is taken from Ref. [6]

³⁾ The properties of the diagonal and off-diagonal vortices in the $SU(2)$ spin model are very similar to the properties of, respectively, the monopoles and minipoles in the MaA projection of the $SU(2)$ gluodynamics, respectively [14].

The generalization of the MaA projection to the $SU(N)$ spin models is quite straightforward. The maximization functional is

$$R[U] = \sum_{\mathbf{x}} \sum_h \text{Tr} \left(U_{\mathbf{x}} \lambda_h U_{\mathbf{x}}^+ \lambda_h \right), \quad (8)$$

where the index h runs over the Cartan subgroup of the $SU(N)$ group and λ_a are the generators of the $SU(N)$ group. The functional R is invariant under $[U(1)]_L^{N-1} \times [U(1)]_R^{N-1}$ global symmetry group. Gauge fixed partition function has the form (4) where one has to take the corresponding $SU(N)$ action and the functional $R[U]$. Now, there are $N(N+1)/2 - 1$ independent abelian spin fields: $N - 1$ diagonal fields and $N(N - 1)/2$ off-diagonal spin fields. Therefore there are $(N - 1)$ species of the diagonal vortices and $N(N - 1)/2$ species of the off-diagonal vortices, respectively. In analogy with $SU(2)$ spin model one can expect that the phase transition in the $SU(N)$ spin model is accompanied by the condensation of the $N - 1$ species of the diagonal vortices. The off-diagonal vortices should show a random distribution and should not be relevant for mass gap generation in the $SU(N)$ spin model.

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