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GRAVITY OF MONOPOLE AND STRING AND GRAVITATIONAL CONSTANT IN ^{3}HE-A

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We discuss the effective metric produced in superfluid ^{3}He-A by such topological objects as the radial disgyration and monopole. In relativistic theories these metrics are similar to that of the local string and global monopole correspondingly. But in ^{3}He-A they have the negative angle deficit, which corresponds to the negative mass of the topological objects. The effective gravitational constant in superfluid ^{3}He-A, deduced from the comparison with relativistic theories, is $G \sim \Delta^{-2}$, where the gap amlitude Δ plays the part of the Planck energy. G depends on temperature roughly as $(1-T^2/T_c^2)^{-2}$ and corresponds to the screening of the Newton's constant.

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Introduction. The effective gravity arises in many condensed matter systems. The typical examples are the crystal with dislocations and disclinations, which models the effective space with curvarture and torsion (see the References in the latest papers [1, 2] on this subject), and the normal (or superfluid) liquid where the sound waves (or phonons) propagate in the effective Lorentzian space generated by the background (super) flow [3, 4]. However it appears that the superfluid ^{3}He-A provides the most adequate analogies for the relativistic models of the effective gravity, which allow to simulate many different properties of the quantum vacuum [5 – 8].

The quasiparticles in ^{3}He-A are chiral and massless fermions. Their spectrum is determined by 3 parameters. One of them is the gap amplitude $\Delta(T)$ which plays the part of the Planck cut-off energy. Below this cut-off, which depends on temperature T, the fermions are "relativistic" with the spectrum:

$$E^2(\mathbf{k}) + g^{ik}(k_i - eA_i)(k_k - eA_k) = 0. \quad (1)$$

Here $\mathbf{A}$ is the dynamical vector potential of the induced "electromagnetic field", $\mathbf{A} = k_F\hat{\mathbf{l}}$, where $\hat{\mathbf{l}}$ is unit vector in the direction of the gap nodes in the momentum space. The same vector determines the uniaxial anisotropy of the metric tensor of the effective space which governs the motion of fermions. In equilibrium this metric is

$$g^{ik} = -c_\perp^2(\delta^{ik} - l^i l^k) - c_\parallel^2 l^i l^k, \quad g^{00} = 1. \quad (2)$$

Here $c_\perp = \Delta(T)/p_F$ and $c_\parallel = v_F$ (with $c_\perp \ll c_\parallel$) are the "speeds of light" propagating transverse to $\hat{l}$ and along $\hat{l}$ correspondingly. The Fermi momentum k_F and the Fermi velocity v_F are practically independent from T while $\Delta(T)$ strongly depends on T: $\Delta^2(T) \sim \Delta^2(0)(1 - T^2/T_c^2)$, where $T_c \sim \Delta(0)$ is the temperature of superfluid transition.

If the $\hat{l}$-field is homogeneous, say, $\hat{l} = \hat{z}$, the anisotropy of the metric in Eq.(2) can be removed by rescaling: $z = c_\parallel Z$, $x = c_\perp X$ and $y = c_\perp Y$. However if $\hat{l}$-field is inhomogeneous, the effective metric gains the curvature and the rescaling can be made locally but not globally. This influences the dynamics of the fermions propagating in the texture in the manner similar to the gravitational field.

The question arises what and how big is the analog of the gravitational constant G in ^{3}He-A and what is its temperature dependence. Since in ^{3}He-A the analog of the Planck energy scale is played by gap amplitude $\Delta(T)$ (see Review [8]), the gravitational constant is of order $G \sim 1/\Delta^2(T)$. However the quantitative evaluation of G is not straghtforward: because of the double role of the $\hat{l}$-field, which produces both the "electromagnetic" and "gravitational" effective fields, it is not easy to separate the "electromagnetic" and "gravitational" terms in the ^{3}He-A Lagrangian. The separation can be made only for specific situations. For example, in the discussion of the effect of axial anomaly on the transformation of the fermionic charge into magnetic field and back, only the electromagnetic part of the $\hat{l}$ action was involved [8]. Here we discuss the opposite case when the "electromagnetic" effects of the $\hat{l}$-field are absent, and one has a pure "gravitational" field. This happens if the "magnetic" field $\mathbf{B} = \nabla \times \mathbf{A}$ is absent. The radial disgyration (string) with $\hat{l} = \hat{\rho}$ and the point monopole with $\hat{l} = \hat{r}$ are such textures, since both have $\nabla \times \hat{l} = 0$. Here $\hat{z}, \hat{\rho}, \hat{\phi}$ and $\hat{r}, \hat{\theta}, \hat{\phi}$ are the unit vectors of the cylindrical and spherical coordinate systems correspondingly. Considering the energy of the radial disgyration and the metric produced by this topological singularity, one can deduce the effective gravitational constant G, which couples these two quantities. Another estimation of G follows from the consideration of the so-called clapping mode of the order parameter, which is the analog of graviton.

Conical singularity with negative angle deficit. The radial disgyration is one of the topologically stable linear defects in ^{3}He-A. This is an axisymmetric distribution of the $\hat{\mathbf{l}}$ vector

$$\hat{l}(\mathbf{r}) = \hat{\rho}, \tag{3}$$

with the axis of the defect line along $\hat{z}$. The interval corresponding to the metric in Eq.(2) is

$$ds^2 = dt^2 - \frac{1}{c_\perp^2} dz^2 - \frac{1}{c_\parallel^2}\left(dr^2 + \frac{c_\parallel^2}{c_\perp^2} r^2 d\phi^2\right), \tag{4}$$

Rescaling the radial and axial coordinates $\rho = c_\parallel R$, $z = c_\perp Z$ one obtains

$$ds^2 = dt^2 - dZ^2 - dR^2 - a^2 R^2 d\phi^2, \quad a^2 = c_\parallel^2/c_\perp^2 > 1. \tag{5}$$

In relativistic theories such metric, but with $a^2 < 1$, arises outside the local strings. The space outside the string core is flat, but the proper length $2\pi R a$ of the circumference of radius R around the axis is smaller than $2\pi R$, if $a < 1$. In our case we have $a^2 > 1$, i.e. the "negative angle deficit". The conical singularity gives rise to the curvarture which is concentrated at the axis of disgyration ($R = 0$) [9, 10]:

$$\mathcal{R}^{R\phi}_{R\phi} = 2\pi \frac{a-1}{a} \delta_2(\mathbf{R}), \quad \delta_2(\mathbf{R}) = \delta(X)\delta(Y). \tag{6}$$

Such metric can arise from the Einstein equations for the local cosmic string with the singular energy density concentrated in the string core

$$\mathcal{T}^0_0 = \frac{1-a}{4Ga} \delta_2(\mathbf{R}). \tag{7}$$

where G is the gravitational constant. Since $a = c_{\parallel}/c_{\perp} \gg 1$, this should be rather unusual cosmic string with a large negative mass of Planck scale. If one finds such singular contribution to the energy density of ^{3}He-A in the presence of radial disgyration one can identify the effective gravitational constant in ^{3}He-A.

Let us consider the distribution of the (orbital) order parameter – the complex vector $\mathbf{e}$ – in the radial disgyration:

$$\mathbf{e}(\mathbf{r}) = f(\rho)\hat{\phi} + i\hat{z}, \quad f(\rho = 0) = 0, \quad f(\rho = \infty) = 1. \tag{8}$$

This order parameter influences the energy spectrum of the fermions in such a way, that it is equivalent to the effective metric

$$ds^2 = dt^2 - dZ^2 - dR^2 - a^2 f^{-2}(Rc_{\parallel})R^2 d\phi^2. \tag{9}$$

The function $f(\rho)$ can be obtained from the Ginzburg-Landau free energy functional, Eq.(5.4) + Eq.(7.17) in [11], which for the chosen Ansatz Eq.(8) has the form

$$F = K\frac{v_F k_F^2}{96\pi^2}\int_0^{z_0} dz \left\{ \int_{\rho<\rho_0} d^2\rho \left[\Lambda(1-f^2)^2 + \frac{f^2}{\rho^2} + \left(\frac{df}{d\rho}\right)^2 \right] - 2\pi \int_0^{\rho_0} d\rho \frac{df^2}{d\rho} \right\}. \tag{10}$$

Here ρ_0 and z_0 are the radius and the height of the cylindrical vessel with the disgyration on the axis; $\Lambda \sim \Delta^2/v_F^2$; the overall dimensionless factor K in the Ginzburg-Landau region close to the transition temperature T_c is

$$K(T) = 1 - T^2/T_c^2, \quad T \to T_c. \tag{11}$$

One can see that the first term in the curly brackets in Eq.(10) is some kind of the dilaton field, while the second term is just what we need: it is the pure divergence and thus can be represented as the singular term, which does not depend on the exact structure of the disgyration core, but nevertheless contributes the core energy:

$$\mathcal{F}_{div} = -2\pi K\frac{v_F k_F^2}{96\pi^2}\delta_2(\rho), \quad F_{div} = \int d^3x \mathcal{F}_{div} = -K\frac{v_F k_F^2}{48\pi}. \tag{12}$$

Now let us extract the constant G by comparing this core energy with the string mass M obtained by integration of $\mathcal{T}_0^0$:

$$M = \int d^3X\sqrt{-g}\mathcal{T}_0^0 = \frac{1-a}{4G}Z_0. \tag{13}$$

Translating this to the ^{3}He-A language, where the "proper" length is $Z_0 = z_0/c_{\perp}$, and taking into account that $a = c_{\parallel}/c_{\perp} \gg 1$ one has

$$M = -\frac{c_{\parallel}}{4Gc_{\perp}^2}z_0. \tag{14}$$

Then from equation, $F_{div} = M$, one obtains the gravitational constant

$$G(T) = 12\pi/K(T)\Delta^2(T). \tag{15}$$

Though we cannot extrapolate the temperature dependence of $K(T)$ in Eq.(11) to the low T, we can expect that the overall temperature dependence can be approximated by

$$G(T) \sim G(0)\left(1 - \frac{T^2}{T_c^2}\right)^{-2}, \; G(0) \sim \frac{1}{T_c^2} \sim \frac{1}{\Delta^2(0)}, \tag{16}$$

where $G(0)$ is the value of G at $T=0$.

The negative mass M does not mean that the vacuum is unstable towards formation of the string: the energy of the radial disgyration is dominated by the first (dilatonic) term in the curly brackets in Eq.(10)

$$E_{disg}(\rho_0) = K\frac{v_F k_F^2}{48\pi} z_0 \ln\frac{\rho_0 \Delta}{c_\parallel}. \quad (17)$$

Translating this to the relativistic language one obtains

$$E_{disg}(R_0) = \frac{a}{4G} Z_0 \ln\frac{R_0}{R_{Planck}}, \quad (18)$$

where $R_{Planck} = 1/\Delta$ is the Planck radius. This energy is larger by the logarithmical factor than the negative "mass of matter" $F_{\rm div} = M$ in Eq.(13) related to the string core. So the situation in this example of the ^{3}He-A gravity is as follows: The energy in Eq.(17) is not gravitating, but it determines the metric. This metric through the Einstein equation gives rise to the negative mass M, which then contributes to the core energy of disgyration. This is an example of how in the effective theory the vacuum is not gravitating but determines the metric.

Gravitational constant from graviton energy-momentum. An independent estimation of G in ^{3}He-A is obtained using the energy of the graviton field. In the relativistic theory the energy density of the graviton propagating along Z is

$$T_0^0 = \frac{1}{16\pi G}\left[(\partial_Z h_{XY})^2 + \frac{1}{4}((\partial_Z (h_{XX} - h_{YY}))^2\right]. \quad (19)$$

Let us consider the corresponding energy density in ^{3}He-A. For this purpose we choose the complex order parameter vector $\mathbf{e}$ in the form

$$\mathbf{e}(z) = \left[\left(1+\frac{1}{2}h_{XX}(z)\right)\hat{x} + \frac{1}{2}h_{XY}(z)\hat{y}\right] + i\left[\left(1+\frac{1}{2}h_{YY}(z)\right)\hat{y} + \frac{1}{2}h_{XY}(z)\hat{x}\right]. \quad (20)$$

It corresponds to the following effective metric for the quasiparticles propagating on the background of this order parameter

$$ds^2 = dt^2 - dZ^2 - (1+h_{XX}(Z))dX^2 - (1+h_{YY}(Z))dY^2 - 2h_{XY}(Z)dXdY. \quad (21)$$

The gradient part of the Ginzburg-Landau free energy functional for this Ansatz, Eq.(20), has the form

$$\mathcal{F} = K\frac{v_F k_F^2}{96\pi^2}\left(\nabla_i \mathbf{e} \nabla_i \mathbf{e}^* + \nabla_i e_j \nabla_j e_i^* + \nabla_i e_i \nabla_j e_j^*\right), \quad (22)$$

$$\mathcal{F} = K\frac{v_F k_F^2}{192\pi^2}\left[\frac{1}{4}((\partial_z (h_{XX} + h_{YY}))^2 + (\partial_Z h_{XY})^2 + \frac{1}{4}((\partial_z (h_{XX} - h_{YY}))^2\right]. \quad (23)$$

The first term in Eq.(23) describes the so-called pair-breaking mode of the order parameter, which is the analog of the (spin 0) dilaton energy. The other two terms describe the so-called clapping mode, which corresponds to the (spin 2) graviton [7]. From the comparison of the clapping mode energy in Eq.(23) and the graviton energy in Eq.(19), one obtains the same value for the gravitational constant as in Eq.(15).

Monopole. In ^{3}He-A the monopole is the hedgehog in the $\hat{l}$-field, $\hat{l} = \hat{r}$, which is the termination point of the quantized vortex (the vortex is analogous to the spinning or torsion string). The effective metric far from the string is

$$ds^2 = \left(dt + \frac{\hbar}{2m_3 c_\perp^2}(1 - \cos\theta)d\phi\right)^2 - \frac{1}{c_\parallel^2}dr^2 - \frac{1}{c_\perp^2}r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (24)$$

The spinning string terminating on the monopole can be removed by the following trick. Let us introduce the electrically charged ^{3}He-A, i.e. the superconductor with the ^{3}He-A order parameter. Then put the t'Hooft – Polyakov magnetic monopole to the hedgehog. In this case the Abrikosov string will be cancelled and thus one obtains only the point singularity in the $\hat{l}$-field – the hedgehog – with pinned magnetic monopole [12]. After rescaling of the radial coordinate $R = r/c_\parallel$ one obtains the metric

$$ds^2 = dt^2 - dR^2 - a^2R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad a^2 = c_\parallel^2/c_\perp^2. \quad (25)$$

This metric describes the 3D conical singularity [9]. In relativistic theories such metric, but with $a^2 < 1$, arises for the global monopoles [13, 14]. In our case we have $a^2 > 1$, i.e. the "negative deficit" of the solid angle. Such situation is also different from the solid angle deficit $> 4\pi$ discussed in [15], where $a^2 < 0$ giving rise to instability of the stationary monopole and to the inflation. The nonzero curvarture elements and the corresponding energy density of matter are [9]

$$\mathcal{R}^{\theta\phi}_{\theta\phi} = -\frac{1 - a^2}{a^2R^2}, \; \mathcal{R} = 2\mathcal{R}^{\theta\phi}_{\theta\phi}, \quad \mathcal{T}^0_0 = \frac{1 - a^2}{a^2 8\pi G R^2}. \quad (26)$$

Integrating the energy density one obtains the negative contribution to the monopole energy (at $a \gg 1$):

$$M(R_0) = \int_{R<R_0} d^3R\sqrt{-g}\mathcal{T}^0_0 = -\frac{a^2}{2G}R_0 \quad (27)$$

Translating to the ^{3}He-A language with the value of G from Eq.(15) and with $r_0 = R_0 c_\parallel$ being the radius of spherical vessel one obtains

$$M(r_0) = -\frac{c_\parallel}{2c_\perp^2 G}r_0 = -\frac{K(T)}{24\pi}v_F k_F^2 r_0. \quad (28)$$

The real energy of the radial $\hat{l}$-texture (without the attached spinning string) is $2|M(r_0)|$ as follows from Eq.(22). However we cannot unambiguously identify the obtained negative mass with some specific term in ^{3}He-A: all the terms have the same $1/r^2$-dependence. This is distinct from the case of disgyration, where the negative contribution has the δ-function singularity and we could easily identify it with the similar negative-energy δ-function contribution in the ^{3}He-A action. One possibility, that this negative mass can be identified as the interaction of the magnetic field of the monopole with the orbital momentum of $\hat{l}$-field in the charged ^{3}He-A, will be discussed elsewhere.

Conclusion. We considered two metrics with nonzero curvature arising in the ^{3}He-A textures. The same metrics occur outside the local cosmic string and the global cosmic monopole, both with the negative mass. In the case of the local cosmic string the negative energy comes from the δ-function singularity of the curvarture. For the ^{3}He-A disgyration the negative energy contribution to the textural energy also comes from δ-function term in the action. Identifying these two negative energy terms, we obtained the value of the effective gravitational constant $G(T)$ in ^{3}He-A. The same value is obtained from the energy-momentum tensor for the analog of the graviton in ^{3}He-A. $G(T)$ is inversely proportional to the square of the "Planck" energy and depends on T increasing with T, which corresponds to the vacuum screening of the gravity. The temperature dependence of the gravitational constant leads to its time dependence during the evolution of the Universe. The latter has been heavily discussed starting with the Dirac proposal (see Review[16]).

From the ^{3}He-A consideration it appeared that there are two contributions to the temperature dependence of G in Eq.(15). (i) The dependence, which comes from the factor $K(T)$, is the traditional one. Since the effective gravity is obtained from the integration over the fermionic (or bosonic) degrees of freedom [17], it is influenced by the thermal distribution of fermions. In relativistic theories, even at low T the renormalization of G is model-dependent: it depends not only on the fermionic and bosonic fields, but also on the cut-off function if the gravitons are included[18]. (ii) Another source of the temperature dependence of $G(T)$ in Eq.(15) is that the "Planck" energy cut-off $\Delta(T)$ depends on temperature. Even at low T this dependence is determined by the transPlanckian physics.

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1. M.O.Katanaev and I.V Volovich, *Scattering on Dislocations and Cosmic Strings in the Geometric Theory of Defects*, gr-qc/9801081.
2. R.Bausch and R.Schmitz, Phys. Rev. Lett. **80**, 2257 (1998).
3. W.G.Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
4. M.Visser, *Acoustic black holes: horizons, ergospheres, and Hawking radiation*, gr-qc/9712010.
5. G.E.Volovik and T.Vachaspati, Int. J. Mod. Phys. **B10**, 471 (1996).
6. T.Jacobson and G.E.Volovik, *Event horizons and ergoregions in ^{3}He*, cond-mat/9801308.
7. G.E.Volovik, *Exotic properties of superfluid ^{3}He*, World Scientific, Singapore - New Jersey - London - Hong Kong, 1992.
8. G.E.Volovik, *Axial anomaly in 3He-A*, cond-mat/9802091.
9. D.D.Sokolov, A.A.Starobinsky, Doklady AN SSSR, **234**, 1043 (1977) [Sov. Phys. - Doklady, **22**, 312 (1977)].
10. M.Banados, C.Teitelboim, and J.Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
11. D.Vollhardt and P.Wölfle, *The superfluid phases of helium 3*, Francis & Taylor, London, 1990.
12. G.E.Volovik, Usp. Fiz. Nauk. **143**, 73 (1984); [Soviet Phys. Usp. **27**, 363 (1984)].
13. M.Barriola and A.Vilenkin, Phys. Rev. Lett. **63**, 341 (1989).
14. D.Harari and C.Lousto, Phys. Rev. **D42**, 2626 (1990).
15. I.Cho and A.Vilenkin, Phys.Rev. **D56**, 7621 (1997).
16. J.D.Barrow, *Varying G and Other Constants*, gr-qc/9711084.
17. A.D.Sakharov, Sov. Phys. Doklady **12**, 1040 (1968).
18. M.Reuter, Phys. Rev. **D57** 971 (1998).