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VORTEX VS SPINNING STRING: IORDANSKII FORCE AND GRAVITATIONAL AHARONOV-BOHM EFFECT.

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We discuss the transverse force acting on the spinning cosmic string, moving in the background matter. It comes from the gravitational Aharonov – Bohm effect and corresponds to the Iordanskii force acting on the vortex in superfluids, when the vortex moves with respect to the normal component of the liquid.

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Introduction. In superfluids, with their two-fluid hydrodynamics (for superfluid and normal components of the liquid) there are 3 different topological contributions to the force acting on the quantized vortex [1]. The more familiar Magnus force arises when the vortex moves with respect to the superfluid vacuum. For the relativistic cosmic string such force is absent since the corresponding superfluid density of the quantum physical vacuum is zero. However the analog of this force appears if the cosmic string moves in the uniform background charge density [2, 3]. The other two forces of topological origin also have analogs for the cosmic strings: one of them comes from the analog of the axial anomaly in the core of electroweak string (see Reviews [4]), and another one – the Iordanskii force – is now under active discussion in condensed matter community [5 – 7].

As distinct from the Magnus force, the Iordanskii force [8, 9] arises when the vortex moves with respect to the heat bath represented by the normal component of the liquid, which consists of the quasiparticle excitations. The latter corresponds to the matter in particle physics. The interaction of quasiparticles with the velocity field of the vortex resembles the interaction of the matter with the gravitational field induced by such cosmic string, which has an angular momentum, – the so-called spinning cosmic string [10]. The spinning string induces the peculiar space-time metric, which leads to the time delay for any particle orbiting around the string with the same speed, but in opposite directions [11]. This gives rise to the quantum gravitational Aharonov – Bohm (AB) effect [10, 12, 13]. We discuss here how the same effect leads to the asymmetry in the scattering of particles on the spinning string and finally to the Iordanskii lifting force acting on the spinning string.

Vortex vs spinning cosmic string. To clarify the analogy between the Iordanskii force and AB effect, let us consider the simplest case of phonons propagating in the velocity field of the quantized vortex in the Bose superfluid ^{4}He. According to the Landau theory of superfluidity, the energy of quasiparticle moving in the superfluid velocity field $\mathbf{v}_s(\mathbf{r})$ is Doppler shifted: $E(\mathbf{p}) = \epsilon(\mathbf{p}) + \mathbf{p}\cdot\mathbf{v}_s(\mathbf{r})$. In the case of the phonons with the spectrum $\epsilon(\mathbf{p}) = cp$, where c is the sound velocity, the energy-momentum relation is thus

$$\left(E - \mathbf{p}\cdot\mathbf{v}_s(\mathbf{r})\right)^2 = c^2p^2 \ . \tag{1}$$

The Eq.(1) can be written in the general Lorentzian form with $p_\mu = (E, \mathbf{p})$:

$$g^{\mu\nu}p_\mu p_\nu = 0 \ , \ g^{00} = 1, \quad g^{0i} = -v_s^i, \ \ g^{ik} = -c^2\delta^{ik} + v_s^i v_s^k \ . \tag{2}$$

Thus the dynamics of phonons in the presence of the velocity field is the same as the dynamics of photons in the gravity field [14]: both are described by the lgiht-cone equation $ds = 0$, where the interval ds is given by the inverse metric $g_{\mu\nu}$:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \ . \tag{3}$$

Here we are interested in the velocity field circulating around quantized vortex, $\mathbf{v}_s = N\kappa\hat{\phi}/2\pi r$, where κ is the quantum of circulation and N is the circulation quantum number. This flow induces the effective space, where the phonon is propagating along geodesic curves, with the interval

$$ds^2 = \left(1 - \frac{v_s^2}{c^2}\right)\left(dt + \frac{N\kappa d\phi}{2\pi(c^2 - v_s^2)}\right)^2 - \frac{dr^2}{c^2} - \frac{dz^2}{c^2} - \frac{r^2 d\phi^2}{c^2 - v_s^2} \ . \tag{4}$$

Far from the vortex, where v_s^2/c^2 is small and can be neglected, one has

$$ds^2 = \left(dt + \frac{d\phi}{\omega}\right)^2 - \frac{1}{c^2}(dz^2 + dr^2 + r^2 d\phi^2) \ , \ \omega = \frac{2\pi c^2}{N\kappa}. \tag{5}$$

The connection between the time and the azimuthal angle ϕ in the interval suggests that there is a characteristic angular velocity ω. For the vortex in superfluid ^{4}He, where $\kappa = 2\pi\hbar/m_4$ and m_4 is the mass of ^{4}He atom, it is $\omega = m_4 c^2/N\hbar$. The similar metric with rotation was obtained for the so-called spinning cosmic string in 3 + 1 space-time, which has the rotational angular momentum J concentrated in the string core, and for the spinning particle in the 2+1 gravity [10, 13, 15, 16]:

$$ds^2 = \left(dt + \frac{d\phi}{\omega}\right)^2 - \frac{1}{c^2}(dz^2 + dr^2 + r^2 d\phi^2) \ , \ \omega = \frac{1}{4JG}, \tag{6}$$

where G is the gravitational constant. This gives the following correspondence between the circulation $N\kappa$ around the vortex and the angular momentum J of the spinning string

$$\kappa N = 8\pi JG \ . \tag{7}$$

Though we consider the analogy between the spinning string and vortices in superfluid ^{4}He, there is a general statement that vortices in any superfluids have the properties of the spinning cosmic strings [2]. In particular, the spinning string generates the density of the angular momentum in the vacuum outside the string [17]. The density of the angular momentum in the superfluid vacuum outside the vortex is also nonzero and equals $\hbar Nn$, where n is the number density of elementary bosons in superfluid vacuum: the density of ^{4}He atom in superfluid ^{4}He or of Cooper pairs in superfluid ^{3}He.

Gravitational AB effect. The effect peculiar for the spinning string, which can be modelled in condensed matter, is the gravitational AB topological effect [10]. On the classical level the propagation of phonons is described by the equation $ds^2 = 0$. Outside the string the metric, which enters the interval ds, is locally flat. But there is the time difference for the particles propagating around the spinning string in the opposite directions. For the vortex (at large distances from the core) this time delay approaches [11]

$$2\tau = \frac{4\pi}{\omega} \ . \tag{8}$$

This asymmetry between the particles moving on different sides of the vortex is the origin of the Iordanskii force acting on the vortex in the presence of the net momentum of the quasiparticles. On the quantum level, the connection between the time variable t and the angle variable ϕ in the metric Eq.(6) implies that the scattering cross section of phonons (photons) on the vortex should be the periodic function of the energy with the period equal to $\hbar\omega$. The asymmetric part of this cross section gives rise to the Iordanskii force.

There was an extreme interpretation of the gravitational AB effect put forward by Mazur [10]. He argued that for the infinitely thin spinning cosmic string there is a region where the causality is violated. To avoid causality violation the string should be transparent for the excitations and this is possible only if in the presence of the spinning cosmic string the energy of the elementary particles is strictly quantized: $E = n\hbar\omega$. In other words the gravitational AB effect leads to the energy quantization in the same manner as the quantization of the electric charge should take place in the presence of the Dirac magnetic monopole. On the other hand there are solutions for cosmic strings, which do not contain the closed timelike curves [18]. In this case the severe energy quantization is not necessary, see however discussion in [12, 13, 19, 20]. In any case the periodicity with the period $\Delta E = \hbar\omega$ is retained and the symmetric part of the scattering cross section of particle with energy E in the background of spinning string with zero mass is [12, 13]:

$$\frac{d\sigma}{d\theta} = \frac{\hbar c}{2\pi E \sin^2(\theta/2)} \sin^2 \frac{\pi E}{\hbar\omega} \ . \tag{9}$$

We argue that in addition to this symmetric part there is the topological asymmetric contribution, which gives rise to transverse cross section

$$\sigma_\perp = \int_0^{2\pi} d\theta \ \sin\theta \ |a(\theta)|^2. \tag{10}$$

The asymmetry in the scattering of the quasiparticles on the velocity field of the vortex has been calculated by Sonin for phonons and rotons in ^{4}He [5] and by Cleary [21] for the Bogoliubov-Nambu quasiparticles in conventional superconductors. In the case of phonons the propagation is described by the Lorentzian equation for the scalar field, $g^{\mu\nu}\partial_\mu\partial_\nu \ \Phi = 0$. In the asymptotic region the quadratic terms $\mathbf{v}_s^2/c^2$ can be neglected and this equation can be rewritten as [5]

$$E^2\Phi - c^2\left(-i\nabla + \frac{E}{c}\mathbf{v}_s(\mathbf{r})\right)^2 \Phi = 0 \ . \tag{11}$$

This equation maps the problem under discussion to the Aharonov-Bohm problem for the magnetic flux tube [22] with the vector potential $\mathbf{A} = \mathbf{v}_s$, where the electric charge e is substituted by the mass E/c^2 of the particle [12, 17, 23]. Because of the mapping between the electric charge and the mass of the particle, the Lorentz force, which acts on the flux tube in the presence of electric current, has its counterpart – the Iordanskii force, which acts on the vortex in the presence of the mass current carried by the normal component [7].

If one directly follows the mapping of the phonon scattering on vortices described by Eq.(11) to the AB scattering, one obtains the AB result [22] for the symmetric part of the differential cross section, now written in the form of Eq.(9). There is a not very important difference, which comes from the definition of the quasiparticle current: as distinct from the charged particles in the AB effect, the current in our case is not gauge invariant. As a result the scattering of the phonon with momentum p and with the energy E by the vortex is somewhat different [5]:

$$\frac{d\sigma}{d\theta} = \frac{\hbar c}{2\pi E} \cot^2 \frac{\theta}{2} \ \sin^2 \frac{\pi E}{\hbar\omega} \ . \tag{12}$$

The difference between Eq.(12) and the AB result Eq.(9) is $(c/2\pi E)\ \sin^2(\pi E/\omega)$, which is independent of the scattering angle θ and thus is not important for the singularity at small scattering angles. For small E the result in Eq.(12) was obtained by Fetter [24]. The generalization of the Fetter result for the quasiparticles with arbitrary spectrum $\epsilon(\mathbf{p})$ (rotons in ^{4}He and the Bogoliubov-Nambu fermions in superconductors) was recently suggested in Ref.[25]: In our notations it is $(N\kappa^2 p/8\pi v_G^2)\cot^2(\theta/2)$, where $v_G = d\epsilon/dp$ is the group velocity of quasiparticle.

Asymmetric cross section. The Lorentz-type Iordanskii force comes from the asymmetric singularity in the cross section [7]. This additional topological term is determined by the same asymptote of the flow velocity, which causes singularity at the small angles in the symmetric cross section. The asymmetric part of the differential cross section gives the following transverse cross section [5]

$$\sigma_\perp = \frac{\hbar}{p} \ \sin \frac{2\pi E}{\hbar\omega}. \tag{13}$$

At low E this result was generalized for arbitrary excitations with the spectrum $E(\mathbf{p}) = \epsilon(\mathbf{p}) + \mathbf{p}\cdot\mathbf{v}_s$ moving in the background of the velocity field $\mathbf{v}_s$ around the vortex, using a simple classical theory of scattering [5]. Far from the vortex, where the circulating velocity is small, the trajectory of the quasiparticle is almost the straight line parallel, say, to the axis y, with the distance from the vortex line being the impact parameter x. It moves along this line with the almost constant momentum $p_y \approx p$ and almost constant group velocity $v_G = d\epsilon/dp$. The change in the transverse momentum during this motion is determined by the Hamiltonian equation $dp_x/dt = -\partial E/\partial x = -p_y \partial v_{sy}/\partial x$, or $dp_x/dy = -(p/v_G)\partial v_{sy}/\partial x$. The transverse cross section is obtained by integration of $\Delta p_x/p$ over the impact parameter x:

$$\sigma_\perp = \int_{-\infty}^{+\infty} \frac{dx}{v_G} \int_{-\infty}^{+\infty} dy \frac{\partial v_{sy}}{\partial x} = \frac{N\kappa}{v_G} \ . \tag{14}$$

Note that this result is a pure classical: the Planck constant $\hbar$ drops out.

Iordanskii force on spinning string. This asymmetric part of scattering, which describes the momentum transfer in the transverse direction, after integration over the distribution of excitations gives rise to the transverse force acting on the vortex if the vortex moves with respect to the normal component. This is the Iordanskii force:

$$\mathbf{f}_{\rm Iordanskii} = \int \frac{d^3p}{(2\pi)^3} \sigma_\perp(p) v_G n(\mathbf{p}) \mathbf{p} \times \hat{\mathbf{z}} = -N\kappa \hat{\mathbf{z}} \times \int \frac{d^3p}{(2\pi)^3} n(\mathbf{p})\mathbf{p} = N\kappa \mathbf{P}_n \times \hat{\mathbf{z}}. \tag{15}$$

It depends only on the density of mass current $\mathbf{P}_n$ carried by excitations (matter) and on the circulation $N\kappa$ around the vortex. This confirms the topological origin of this force. In the case of the equilibrium distribution of quasiparticles one has $\mathbf{P}_n = \rho_n \mathbf{v}_n$, where ρ_n and $\mathbf{v}_n$ are the density and velocity of the normal component of the liquid (to avoid the conventional Magnus force, we assume that the asymptotic velocity of the superfluid component of the liquid is zero in the vortex frame).

Since the Eq.(15) was obtained using the asymptotic befavior of the flow field $\mathbf{v}_s$, which induces the same effective metric as the metric around the spinning string, one can apply this result directly to the spinning string. The asymmetric cross-section of the scattering of relativistic particles on the spinning string is given by Eq.(13). This means that in the presence of the momentum of matter the spinning cosmic string experiences the kind of the lifting force, which corresponds to the Iordanskii force in superfluids. This force can be obtained by relativistic generalization of the Eq.(15). The momentum density $\mathbf{P}_n$ of quasiparticles should be substituted by the component T_0^i of the energy-momentum tensor. As a result, for 2+1 space-time and for small energy E, which corresponds to the low temperature T of the matter, the Iordanskii force on spinning string moving with respect to the matter is

$$f^{\alpha}_{\rm Iordanskii} = 8\pi J G e^{\alpha\beta\gamma} u_\beta u_\mu T^{\mu}_{\gamma} \ . \qquad (16)$$

Here u_α is the 3-velocity of the string and T^{μ}_{γ} is the asymptotic value of the energy-momentum tensor of the matter at the spot of the string. Using the Einstein equations one can rewrite this as

$$f^{\alpha}_{\rm Iordanskii} = J e^{\alpha\beta\gamma} u_\beta u_\mu R^{\mu}_{\gamma} \ , \qquad (17)$$

where R^{μ}_{γ} is the Riemannian curvature at the position of the string,which is induced by external sources. This corresponds to the force acting on particle with the spin J from the gravitational field due to interaction of the spin with the Riemann tensor [26].

Conclusion. There is an analogy between the asymptotic velocity field far from the vortex core in superfluids and the gravitational field induced by the spinning cosmic string. As a result both systems experience the gravitational Aharonov-Bohm effect, which in particular leads to the Iordanskii force acting on the vortex (the spinning string), when it moves with respect to the heat bath of quasiparticles (the matter).

Iordanskii force has been experimentally identified in the rotating superfluid ^{3}He-B. According to the theory for the transport of vortices in ^{3}He-B, the Iordanskii force completely determines the mutual friction parameter $d_\perp \approx -\rho_n/\rho$ at low T [27], where ρ is the total density of the liquid. This is in accordance with the experimental data, which show that $d_\perp$ does approach its negative asymptote at low T [28]. At higher T another topological force, which comes from the spectral flow of the fermion zero modes in the vortex core [29, 4], becomes dominating and leads to the sign reversal of $d_\perp$. The observed negative sign of $d_\perp$ at low T provides the experimental verification of the condensed matter analog of the gravitational Aharonov-Bohm effect on spinning cosmic string.

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1. G.E.Volovik, Pis'ma ZhETF **62**, 58 (1995) [JETP Lett. **62**, 65 (1995)].
2. R.L.Davis and E.P.S Shellard, Phys. Rev. Lett. **63**, 2021 (1989).
3. K.-M. Lee, Phys. Rev. **D49**, 4265 (1994).
4. G.E.Volovik and T.Vachaspati, Int. J. Mod. Phys. **B10**, 471 (1996); G.E.Volovik, cond-mat/9802091.
5. E.B.Sonin, Phys. Rev. **B55**, 485 (1997).
6. C.Wexler, Phys. Rev. Lett. **79**, 1321 (1997).
7. A.Shelankov, cond-mat/9802158.
8. S.V.Iordanskii, Ann. Phys. **29**, 335 (1964); ZhETF **49**, 225 (1965); [JETP **22**, 160 (1966)].
9. E.B.Sonin, ZhETF **69**, 921 (1975) [JETP **42**, 469 (1976)].
10. P.O.Mazur, Phys. Rev. Lett. **57**, 929 (1986).
11. D.Harari and A.P. Polychronakos, Phys. Rev. **D38**, 3320 (1988).
12. P.O.Mazur, Phys. Rev. Lett. **59**, 2380 (1987).
13. P.O.Mazur, hep-th/9611206.
14. W.G.Unruh, Phys. Rev. **D14**, 870 (1976); **D51**, 2827 (1995).
15. A.Staruszkievicz, Acta Phys. Polon. **24**, 734 (1963).
16. S.Deser, R.Jackiw, and G. t'Hooft, Ann. Phys. **152**, 220 (1984).
17. B.Jensen and J.Kučera, J. Math. Phys. **34**, 4975 (1993).
18. B.Jensen and H.H.Soleng, Phys. Rev. **D45** 3528 (1992).
19. G.N.Glushchenko, ZhETF **104**, 273 (1995) [JETP **80**, 145 (1995)].
20. G.E.Volovik, Low Temp. Phys. (Kharkov) **24**, 127 (1998).
21. R.M.Cleary, Phys. Rev. **175**, 587 (1968).
22. Y.Aharonov and D.Bohm, Phys. Rev. **115**, 485 (1959).
23. D.V.Gal'tsov and P.S.Letelier, Phys. Rev. **D47**, 4273 (1993).
24. A.L.Fetter, Phys. Rev. **136A**, 1488 (1964).
25. E.Demircan, P.Ao and Q. Niu, Phys. Rev. **B52**, 476 (1995).
26. K.S.Thorne and J.B.Hartle, Phys. Rev. **D31**, 1815 (1985); Y.Mino, M.Sasaki, and T.Tanaka, gr-qc/9705073.
27. N.B.Kopnin, G.E.Volovik, and Ü.Parts, Europhys. Lett. **32**, 651 (1995).
28. T.D.C.Bevan, A.J.Manninen, J.B.Cook, *et al.*, Phys. Rev. Lett. **74**, 750 (1995).
29. T.D.C. Bevan, A.J.Manninen, J.B.Cook, *et al.*, Nature **386**, 689 (1997).