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Low temperature relation for the trace of the energy-momentum tensor in QCD with light quarks

N. O. Agasian[1)]

Institute of Theoretical and Experimental Physics, 117218 Moscow, Russia

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It is shown that the temperature derivatives of the anomalous and normal (quark massive term) contributions to the trace of the energy-momentum tensor in QCD are equal to each other in the low temperature region. The physical consequences of this relation are discussed.

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1. The low-energy theorems, playing an important role in the understanding of the vacuum state properties in quantum field theory, were discovered almost at the same time as quantum field methods have been applied in particle physics (see, for example, Low theorems [1]). In QCD, they were obtained in the beginning of eighties [2]. The QCD low-energy theorems, being derived from the very general symmetry considerations and not depending on the details of confinement mechanism, sometimes give information which is not easy to obtain in another way. Also, they can be used as "physically sensible" restrictions in the constructing of effective theories. Recently, they were generalized to finite temperature and chemical potential case [3, 4]. These theorems were used for investigation of QCD vacuum phase structure in a magnetic field [5] and at finite temperature [6].

The investigation of the vacuum state behavior under the influence of various external factors is known to be one of the central problems of quantum field theory. In the realm of strong interactions (QCD) the main factors are the temperature and the baryon density. At low temperatures, $T < T_c$ (T_c – temperature of the "hadron–quark-gluon" phase transition), the dynamics of QCD is essentially nonperturbative and is characterized by confinement and spontaneous breaking of chiral symmetry (SBCS). In the hadronic phase the partition function of the system is dominated by the contribution of the lightest particles in the physical spectrum. It is well known that due to the smallness of pion mass as compared to the typical scale of strong interactions, the pion plays a special role among other strongly-interacting particles. Therefore for many problems of QCD at zero temperature the chiral limit, $M_\pi \to 0$, is an appropriate one. On the other hand a new mass scale emerges in the physics of QCD phase transitions, namely the critical transition temperature T_c. Numerically the critical transition temperature turns out to be close to the pion mass, $T_c \approx M_\pi$[2)]. However hadron states heavier than pion have masses several times larger than T_c and therefore their contribution to the thermodynamic quantities is damped by Boltzmann factor $\sim \exp\{-M_{hadr}/T\}$. Thus the thermodynamics of the low temperature hadron phase, $T \lesssim M_\pi$, is described basically in terms of the thermal excitations of relativistic massive pions.

In the present paper the low temperature relation for the trace of the energy-momentum tensor in QCD with two light quarks is obtained based on the general dimensional and renormalization-group properties of the QCD partition function and dominating role of the pion thermal exitations in the hadronic phase. The physical consequences of this relation are discussed as well as the possibilities to use it in the lattice studies of the QCD at finite temperature.

2. For non-zero quark mass ($m_q \neq 0$) the scale invariance is broken already at the classical level. Therefore the pion thermal excitations would change, even in the ideal gas approximation, the value of the gluon condensate with increasing temperature[3)]. To determine this dependence use will be made of the general renormalization and scale properties of the QCD partition function.

[1)]e-mail: agasian@heron.itep.ru

[2)]The deconfining phase transition temperature is the one obtained in lattice calculations $T_c(N_f = 2) \simeq 173$ MeV and $T_c(N_f = 3) \simeq 154$ MeV [7].

[3)]At zero quark mass the gas of massless noninteracting pions is obviously scale-invariant and therefore does not contribute to the trace of the energy-momentum tensor and correspondingly to the gluon condensate $\langle (G^a_{\mu\nu})^2 \rangle$.

The QCD Euclidean partition function with two quark flavors has the following form ($\beta = 1/T$)

$$Z = \int [DA] \prod_{q=u,d} [D\bar{q}][Dq] \exp\left\{ -\int_0^\beta dx_4 \int_V d^3x \mathcal{L} \right\}. \quad (1)$$

Here the QCD Lagrangian is

$$\mathcal{L} = \frac{1}{4g_0^2}(G^a_{\mu\nu})^2 + \sum_{q=u,d} \bar{q}[\gamma_\mu(\partial_\mu - i\frac{\lambda^a}{2}A^a_\mu) + m_{0q}]q, \quad (2)$$

where the gauge fixing and ghost terms have been omitted. The free energy density is given by the relation $\beta VF\ (T, m_{0u}, m_{0d}) = -\ln Z$. Eq. (1) yields the following expression for the gluon condensate ($\langle G^2\rangle \equiv \equiv \langle (G^a_{\mu\nu})^2\rangle$)

$$\langle G^2\rangle(T, m_{0u}, m_{0d}) = 4\frac{\partial F}{\partial(1/g_0^2)}. \quad (3)$$

The system described by the partition function (1) is characterized by the set of dimensionful parameters $M, T, m_{0q}(M)$ and dimensionless charge $g_0^2(M)$, where M is the ultraviolet cutoff. On the other hand one can consider the renormalized free energy F_R and by using the dimensional and renormalization-group properties of F_R recast (3) into the form containing derivatives with respect to the physical parameter T and renormalized masses m_q.

The phenomenon of dimensional transmutation results in the appearance of a nonperturbative dimensionful parameter

$$\Lambda = M \exp\left\{ \int_{\alpha_s(M)}^{\infty} \frac{d\alpha_s}{\beta(\alpha_s)} \right\}, \quad (4)$$

where $\alpha_s = g_0^2/4\pi$, and $\beta(\alpha_s) = d\alpha_s(M)/d\ln M$ is the Gell-Mann-Low function. Furthermore, as it is well known, the quark mass has anomalous dimension and depends on the scale M. The renormalization-group equation for $m_0(M)$, the running mass, is $d\ln m_0/d\ln M = -\gamma_m$ and we use the $\overline{MS}$ scheme for which β and γ_m are independent of the quark mass [4, 8]. Upon integration the renormalization-group invariant mass is given by

$$m_q = m_{oq}(M) \exp\left\{ \int^{\alpha_s(M)} \frac{\gamma_{m_q}(\alpha_s)}{\beta(\alpha_s)} d\alpha_s \right\}, \quad (5)$$

where the indefinite integral is evaluated at $\alpha_s(M)$. Next we note that since free energy is renormalization-group invariant quantity its anomalous dimension is zero. Thus F_R has only a normal (canonical) dimension equal to 4. Making use of the renorm-invariance of Λ, one can write in the most general form

$$F_R = \Lambda^4 f\left(\frac{T}{\Lambda}, \frac{m_u}{\Lambda}, \frac{m_d}{\Lambda}\right), \quad (6)$$

where f is some function. From (4), (5) and (6) one gets

$$\frac{\partial F_R}{\partial(1/g_0^2)} = \frac{\partial F_R}{\partial \Lambda}\frac{\partial \Lambda}{\partial(1/g_0^2)} + \sum_q \frac{\partial F_R}{\partial m_q}\frac{\partial m_q}{\partial(1/g_0^2)}, \quad (7)$$

$$\frac{\partial m_q}{\partial(1/g_0^2)} = -4\pi\alpha_s^2 m_q \frac{\gamma_{m_q}(\alpha_s)}{\beta(\alpha_s)}. \quad (8)$$

With the account of (3) the gluon condensate is given by

$$\langle G^2\rangle(T, m_u, m_d) = = \frac{16\pi\alpha_s^2}{\beta(\alpha_s)}\left(4 - T\frac{\partial}{\partial T} - \sum_q (1+\gamma_{m_q}) m_q \frac{\partial}{\partial m_q}\right) F_R. \quad (9)$$

It is convenient to choose such a large scale that one can take the lowest order expressions, $\beta(\alpha_s) \to -b\alpha_s^2/2\pi$, where $b = (11N_c - 2N_f)/3$ and $1 + \gamma_m \to 1$. Thus, we have the following equations for condensates

$$\langle G^2\rangle(T) = = -\frac{32\pi^2}{b}\left(4 - T\frac{\partial}{\partial T} - \sum_q m_q \frac{\partial}{\partial m_q}\right) F_R \equiv -\hat{D}F_R, \quad (10)$$

$$\langle \bar{q}q\rangle(T) = \frac{\partial F_R}{\partial m_q}. \quad (11)$$

3. In the hadronic phase the effective pressure from which one can extract the condensates $\langle\bar{q}q\rangle(T)$ and $\langle G^2\rangle(T)$ using the general relations (10) and (11) has the form

$$P_{eff}(T) = -\varepsilon_{vac} + P_h(T), \quad (12)$$

where $\varepsilon_{vac} = \frac{1}{4}\langle\theta_{\mu\mu}\rangle$ is the nonperturbative vacuum energy density at $T = 0$ and

$$\langle\theta_{\mu\mu}\rangle = -\frac{b}{32\pi^2}\langle G^2\rangle + \sum_{q=u,d} m_q\langle\bar{q}q\rangle \quad (13)$$

is the trace of the energy-momentum tensor. In Eq.(12) $P_h(T)$ is the thermal hadrons pressure. The quark and gluon condensates are given by the equations

$$\langle\bar{q}q\rangle(T) = -\frac{\partial P_{eff}}{\partial m_q}, \quad (14)$$

$$\langle G^2\rangle(T) = \hat{D}P_{eff}, \qquad (15)$$

where the operator $\hat{D}$ is defined by the relation (10)

$$\hat{D} = \frac{32\pi^2}{b}(4 - T\frac{\partial}{\partial T} - \sum_q m_q\frac{\partial}{\partial m_q}). \qquad (16)$$

Consider the $T = 0$ case. One can use the low energy theorem for the derivative of the gluon condensate with respect to the quark mass [2]

$$\frac{\partial}{\partial m_q}\langle G^2\rangle = \int d^4x\langle G^2(0)\bar{q}q(x)\rangle = -\frac{96\pi^2}{b}\langle\bar{q}q\rangle + O(m_q), \qquad (17)$$

where $O(m_q)$ stands for the terms linear in light quark masses.Then one arrives at the following relation

$$\frac{\partial\varepsilon_{vac}}{\partial m_q} = -\frac{b}{128\pi^2}\frac{\partial}{\partial m_q}\langle G^2\rangle + \frac{1}{4}\langle\bar{q}q\rangle =$$

$$= \frac{3}{4}\langle\bar{q}q\rangle + \frac{1}{4}\langle\bar{q}q\rangle = \langle\bar{q}q\rangle. \qquad (18)$$

Note that three fourths of the quark condensate stem from the gluon part of the nonperturbative vacuum energy density. Along the same lines one arrives at the expression for the gluon condensate

$$-\hat{D}\varepsilon_{vac} = \langle G^2\rangle. \qquad (19)$$

In order to get the dependence of the quark and gluon condensates upon T use is made of the Gell-Mann-Oakes-Renner (GMOR) relation ($\Sigma = |\langle\bar{u}u\rangle| = |\langle\bar{d}d\rangle|$)

$$F_\pi^2 M_\pi^2 = -\frac{1}{2}(m_u + m_d)\langle\bar{u}u + \bar{d}d\rangle = (m_u + m_d)\Sigma. \qquad (20)$$

Then we can find the following relations

$$\frac{\partial}{\partial m_q} = \frac{\Sigma}{F_\pi^2}\frac{\partial}{\partial M_\pi^2}, \qquad (21)$$

$$\sum_q m_q\frac{\partial}{\partial m_q} = (m_u + m_d)\frac{\Sigma}{F_\pi^2}\frac{\partial}{\partial M_\pi^2} = M_\pi^2\frac{\partial}{\partial M_\pi^2}, \qquad (22)$$

$$\hat{D} = \frac{32\pi^2}{b}\left(4 - T\frac{\partial}{\partial T} - M_\pi^2\frac{\partial}{\partial M_\pi^2}\right). \qquad (23)$$

Within the described above framework one can derive the thermodynamic relation for the quantum anomaly in the trace of the energy-momentum tensor of QCD. At low temperature the main contribution to the pressure comes from thermal excitations of massive pions. The general expression for the pressure reads

$$P_\pi = T^4\varphi(M_\pi/T), \qquad (24)$$

where φ is a function of the ratio M_π/T. Then the following relation is valid

$$\left(4 - T\frac{\partial}{\partial T} - M_\pi^2\frac{\partial}{\partial M_\pi^2}\right)P_\pi = M_\pi^2\frac{\partial P_\pi}{\partial M_\pi^2}. \qquad (25)$$

With the account of (14), (15), (18), (22) and (25) one gets

$$\Delta\langle\bar{q}q\rangle = -\frac{\partial P_\pi}{\partial m_q}, \quad \Delta\langle G^2\rangle = \frac{32\pi^2}{b}M_\pi^2\frac{\partial P_\pi}{\partial M_\pi^2}, \qquad (26)$$

where $\Delta\langle\bar{q}q\rangle = \langle\bar{q}q\rangle_T - \langle\bar{q}q\rangle$ and $\Delta\langle G^2\rangle = \langle G^2\rangle_T - \langle G^2\rangle$. In view of (22) one can recast (26) in the form

$$\Delta\langle G^2\rangle = -\frac{32\pi^2}{b}\sum_q m_q\Delta\langle\bar{q}q\rangle. \qquad (27)$$

Let us divide both sides of (27) by ΔT and take the limit $\Delta T \to 0$. This yields

$$\frac{\partial\langle G^2\rangle}{\partial T} = -\frac{32\pi^2}{b}\sum_q m_q\frac{\partial\langle\bar{q}q\rangle}{\partial T}. \qquad (28)$$

This can be rewritten as

$$\frac{\partial\langle\theta^g_{\mu\mu}\rangle}{\partial T} = \frac{\partial\langle\theta^q_{\mu\mu}\rangle}{\partial T}, \qquad (29)$$

where

$$\langle\theta^q_{\mu\mu}\rangle = \sum_{q=u,d} m_q\langle\bar{q}q\rangle \text{ and } \langle\theta^g_{\mu\mu}\rangle = (\beta(\alpha_s)/16\pi\alpha_s^2)\langle G^2\rangle$$

are correspondingly the quark and gluon contributions to the trace of the energy-momentum tensor. Note that in deriving this result use was made of the low energy GMOR relation, and therefore the thermodynamic relation (28), (29) is valid in the light quark theory. Thus in the low temperature region when the excitations of massive hadrons and interactions of pions can be neglected, equation (29) becomes a rigorous QCD theorem.

As it was mentioned above the pion plays an exceptional role in thermodynamics of QCD due to the fact that its mass is numerically close to the phase transition temperature while the masses of heavier hadrons are several times larger than T_c. This was the reason we did not consider the role of massive states in the low temperature phase. This question was discussed in detail in Ref.[9]. It was shown there that at low temperatures, the contribution to $\langle\bar{q}q\rangle$ generated by the massive states is very small, less than 5% if T is below 100 MeV. At $T = 150$ MeV, this contribution is of the order of 10%. The influence of thermal excitations of massive hadrons on the properties of the gluon and quark condensates in the framework of the conformal-nonlinear σ-model was also studied in detail in [10].

4. It was shown that the temperature derivatives of the anomalous and normal (quark massive term) contributions to the trace of the energy-momentum tensor in QCD with light quarks are equal to each other in the low temperature region.

Let us consider some physical consequences and possible applications of this relation. To this end we introduce the function

$$\delta_\theta(T) = \frac{\partial}{\partial T}\langle \theta^g_{\mu\mu} - \theta^q_{\mu\mu}\rangle. \quad (30)$$

As it was stated above, the function $\delta_\theta(T)$ at low temperatures is, with good accuracy, close to zero. In the vicinity and at the phase transition point, i.e. in the region of nonperturbative vacuum reconstruction this function changes drastically. To see it, we first consider pure gluodynamics. It was shown in [11] using the effective dilaton Lagrangian, that gluon condensate decreases very weakly with the increase of temperature, up to phase transition point. This result is physically transparent and is the consequence of Boltzmann suppression of thermal glueball excitations in the confining phase.

Further, in Refs.[12] the dynamical picture of deconfinement was suggested based on the reconstruction of the nonperturbative gluonic vacuum. Namely, confining and deconfining phases according to [12] differ first of all in the vacuum fields, i.e., in the value of the gluon condensate and in the gluonic field correlators. It was argued in [12] that color-magnetic (CM) correlators and their contribution to the condensate are kept intact across the temperature phase transition, while the confining color-electric (CE) part abruptly disappears above T_c. Furthermore, there exist numerical lattice measurements of field correlators near the critical transition temperature T_c, made by the Pisa group [13], where both CE and CM correlators are found with good accuracy. These data clearly demonstrate the strong suppression of CE component above T_c and persistence of CM components. Thus, the function $\delta_\theta(T)_{GD} = \partial\langle\theta^g_{\mu\mu}\rangle/\partial T$ can be presented as a δ-function smeared around the critical point T_c with the width $\sim \Delta T$ which defines the fluctuation region of phase transition.

Similar, but more complicated and interesting situation takes place in the theory with quarks. The function $\delta_\theta(T)$ contains the quark term, proportional to the chiral phase transition order parameter $\langle\bar{q}q\rangle(T)$. So it is interesting to check the relation (29) and to study the behavior of the function $\delta_\theta(T)$ in the lattice QCD at finite temperature. It would allow both to test the nonperturbative QCD vacuum at the low temperatures in the confining phase and to extract additional information on the thermal phase transitions in QCD.

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