Pis'ma v ZhETF, vol. 73, iss. 2, pp. 59 – 63

On Compatibility of Tensor Products on Solutions to Commutativity and WDVV Equations

A. Losev[+1)], *I. Polyubin*[*+]

[+]*Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia*

[*]*Landau Institute for Theoretical Physics RAS, 142432 Chernogolovka, Moscow region, Russia*

Submitted 28 December 2000

We show that topological conformal theory contains solutions for WDVV equations for genus zero amplitudes and solutions to Commutativity equations from its quantum mechanics on the annulus. We explain behaviour of these solutions under the tensor product of the theories. We make a conjecture that it is compatible with the construction of solutions to WDVV equations from solution to Commutativity equation and explicitly check it in the first nontrivial case.

PACS: 11.25.Hf

The topological string theory (TST) is based on the 2d conformal topological theory that has nilpotent odd symmetry Q and Q-exact energy-momentum tensor: $T = Q(G)$. The n-point amplitude (traditionally denoted as F) in TST is given as an integral over the moduli space $M_{g,n}$ of complex structures on the genus g Riemann surface Σ_g with n-marked points. The integrand is the correlator of n Q-closed vertex operators $\Phi_{i_1}, \dots, \Phi_{i_n}$ and $2n+2g-6$ superpartners of the energy-momentum tensor G:

$$F_{i_1 \dots i_n} = \int_{M_{g,n}} \langle \Phi_{i_1}(P_1) \dots \Phi_{i_n}(P_n) \times$$

$$\times \prod_{a=1}^{3g+n-3} (\int_\Sigma \mu_a G_L dm^a)(\int_\Sigma \overline{\mu}_a G_R \overline{dm}^a)\rangle \qquad (1)$$

where $G_L (G_R)$ is the superpartner of the energy-momentum tensor for the left (right) movers respectively, μ_a is the basis in the space of Beltrami differentials corresponding to coordinates m^a on the moduli space [1 – 5].

The notion of the amplitude in TST can be extended to the notion of Generalized Amplitude (GA) $GA_{i_1 \dots i_n}$ [6, 7] taking values in cohomologies of the Deligne-Mumford compactification $\overline{M}_{g,n}$ of the moduli space of complex structures. Its value on the cycle $C_A \subset \overline{M}_{g,n}$ is given by the integral

$$GA_{i_1 \dots i_n}(C_A) = \int_{C_A} (\Omega)^{(\dim C_A)}_{i_1 \dots i_n} \qquad (2)$$

where $(\Omega)^{(\dim C_A)}_{i_1 \dots i_n}$ is the $\dim C_A$ component of the closed form $\Omega_{i_1 \dots i_n}$ given by the correlator

$$\Omega_{i_1 \dots i_n} = \langle \Phi_{i_1}(P_1) \dots \Phi_{i_n}(P_n) \times$$

$$\times \exp(\int_\Sigma \sum_a \mu_a G_L dm^a) \exp(\int_\Sigma \sum_{\bar a} \overline{\mu}_{\bar a} G_R d\overline{m}^{\bar a})\rangle. \quad (3)$$

S.Keel [8] found that the homology ring H_* of $\overline{M}_{0,k}$ is generated by cycles numbered by trees. These cycles correspond to the degeneration of a sphere into a collection of spheres joined by the double points with each other. Each sphere correspond to a vertex , point of junction of two spheres corresponds to an edge. Below we will denote such vertices by parenthesis containing labels of points and the labels of double points. For example, cycle that corresponds to degeneration of the sphere with 5 marked points (such that points $1, 2, 3$ belong to the first sphere and points $4, 5$ belong to the second) is denoted as (1236)(745). Here 6 and 7 are labels of the point of junction on the first and the second spheres respectively.

Let us further assume that conformal dimensions of all fields in conformal topological theory is greater or equal then zero, and the only fields that have zero dimensions are vertex operators Φ_i. Then, from the factorization properties of the correlators in conformal field theory we conclude that correlator on the degenerated surface is expressed in terms of correlators on components with insertions of the full basis of vertex operators at the place of the double

[1)]e-mail: lossev@vxitep.itep.ru, polyub@itp.ac.ru

point. For example, the correlator on a surface that is a junction of two spheres looks like:

$$\Omega_{i_1\ldots i_n} = \Omega_{i_1\ldots i_{n_1} j}\Omega_{i_{n_1+1}\ldots i_n k}\eta^{jk} \tag{4}$$

where η^{ij} is inverse to the bilinear pairing on vertex operators (given by a two-point correlator on the sphere) [1, 3, 9].

S. Keel described relations between cycles corresponding to trees. The evaluation of GA on these relations gives a set of constraints that are compactly represented as follows.

Introducing formal parameters T_i, we define the formal series

$$F(T) = \sum_{k=3}^{\infty} \frac{1}{k!} F_{i_1\ldots i_k} T_{i_1}\ldots T_{i_k}. \tag{5}$$

Keel's relations together with the factorization conditions lead to Associativity(WDVV) equations [2, 9, 3, 7] that were extensivly studied by [10]:

$$\frac{\partial^3 F(T)}{\partial T_i \partial T_j \partial T_k}\eta^{kl}\frac{\partial^3 F(T)}{\partial T_l \partial T_p \partial T_q} = \frac{\partial^3 F(T)}{\partial T_i \partial T_p \partial T_k}\eta^{kl}\frac{\partial^3 F(T)}{\partial T_l \partial T_j \partial T_q} \tag{6}$$

We say that the TST(I $\cdot$ II) is a product of theories I and II if its lagrangian is a sum of lagrangians. From this we get:

$$Q^{\mathrm{I\cdot II}} = Q^{\mathrm{I}} + Q^{\mathrm{II}}; G^{\mathrm{I\cdot II}} = G^{\mathrm{I}} + G^{\mathrm{II}}; \Phi_{\mathrm{I}}^{\mathrm{I\cdot II}} = \Phi_i^{\mathrm{I}}\cdot\Phi_{\bar{i}}^{\mathrm{II}} \tag{7}$$

Thus, the differential form $\Omega^{\mathrm{I\cdot II}}_{I_1,\ldots,I_n}$ in the product theory is a pointwise product (on the moduli space of complex structures) of differential forms Ω^{I} and Ω^{II}: $\Omega^{\mathrm{I\cdot II}}_{I_1,\ldots,I_n} = \Omega^{\mathrm{I}}_{i_1\ldots i_n}\Omega^{\mathrm{II}}_{\bar{i}_1\ldots\bar{i}_n}$.

Therefore, the GA for the product of two theories is a product of GA in cohomologies. It could be computed using the intersection product in homologies, in particular, the amplitude equals

$$F^{\mathrm{I\cdot II}}_{I_1,\ldots,I_n} = (F^{\mathrm{I}}\cdot F^{\mathrm{II}})_{I_1,\ldots,I_n} =$$

$$= (N^{-1})^{AB} GA^{\mathrm{I}}_{i_1\ldots i_n}(C_A) GA^{\mathrm{II}}_{\bar{i}_1\ldots\bar{i}_n}(C_B) \tag{8}$$

where C_A is the basis in the space of homologies, $N_{AB} = C_A \cap C_B$. Due to factorization property (4) the GA could be expressed in terms of F, thus (8) could be considered as a formal product of solutions to WDVV equations (even if these solutions are not coming from any TST) [11].

It is believed that the above scheme of reasoning (with some modifications) could be applied to the wider class of theories, for example, to theories that can contain gravitational descendants (that lead to Morita – Mumford classes on the Deligne – Mumford compactification of moduli space). In particular, action in such theory could be deformed by adding to it 2-observables, corresponding to gravitational descendants. As it was shown in [12 – 14, 11] that generalized amplitudes in these theories also could be considered as maps to cohomologies of $\overline{M}_{0,n}$, and , thus, lead to solutions of WDVV equations. So the product operation for GA could be defined in these theories as above.

Each 2d topological theory contains a supersymmetric quantum mechanics whose Hilbert space $\mathcal{H}$ is a space of wave-functions associated to the circle (that are invariant under the rotations of this circle), with supersymmetries and hamiltonian [15, 16]

$$Q_\pm = Q_{L,0} \pm Q_{R,0},\ G_\pm = G_{L,0},\ \pm G_{R,0},\ H = T_{L,0} + T_{R,0}$$

that obey (anti)commutation relations

$$\{Q_+, G_+\} = \{Q_-, G_-\} = H. \tag{9}$$

This quantum mechanics also contains Q_+-closed operators $\phi_i = \int_{x\in S_1}\Phi(x)dx$ where $0 \le x \le 2\pi$ is a standard coordinate on S_1.

We assume that the kernel W of Hamiltonian consists of Q_+ and G_- closed states, and the rest of its eigenvalues have strictly positive real part.

One can show that the regularity of the form $\Omega_{i_1\ldots i_n}$ (3) at points of degeneration of Riemann surface leads to:

$$[\phi_i, \phi_j] = [\phi_i, [G_-, \phi_j]] = 0. \tag{10}$$

Given N operators $\phi_{i_1}, \ldots, \phi_{i_N}$ we define the form $\omega_{i_1,\ldots,i_N}$ on $(C^*)^N/C^*$ taking values in linear operators in W as follows.

Let us parameterize points on C^* by two standard real coordinates $l > 0$ and φ (taking value in the interval $[0, 2\pi]) : z = \exp(l + i\varphi)$. Consider the space of N points on $(C^*)^N$ – it is parameterized by coordinates l_i and φ_i, $i = 1, \ldots, N$.

Let us delete all l-diagonals ($l_i = l_j$ for some pair (i, j)) from the space $(C^*)^N$. The rest of the space is divided into regions labeled by permutations σ of first N integers, such that

$$l_{\sigma(k)} > l_{\sigma(m)}\ iff\ k > m. \tag{11}$$

The form $\omega_{i_1\ldots i_N}$ is defined separately in each region:

$$\omega_{i_1\ldots i_N} = \Pi_W \Phi_{i_{\sigma(1)}} \exp(\hat{H}_{\sigma(1)\sigma(2)})\Phi_{i_{\sigma(2)}} \times$$

$$\times \exp(\hat{H}_{\sigma(2)\sigma(3)})\ldots\Phi_{i_{\sigma(N)}}\Pi_W \tag{12}$$

where

$$\hat{H}_{km} = -H(l_k - l_m) +$$

$$+ G_+(dl_k - dl_m) + G_-(d\varphi_k - d\varphi_m) \qquad (13)$$

and Π_W is the projector on W.

One can easily check that in each region the form is closed: $Q_+\Pi_W = 0$ and

$$d(\hat{H}_{km}) = [Q_+, \hat{H}_{km}]. \qquad (14)$$

Now, we would like to get the closed form on the union of regions: we need to take care about discontinuities. Suppose that at the boundary of the regions $l_k = l_m$. Then, discontinuities in the terms containing $d(l_k - l_m)$ are irrelevant – these terms are closed anyway. So, we have to take care about terms containing no differentials in the $l_k - l_m$ direction and about terms containing $d(\varphi_k - \varphi_m)$ – but these discontinuities are absent due to (10).

Now we come to the issue of the "sausage" compactification: the proper compactification of $(C^*)^N/C^*$ is such that $\omega_{i_1\ldots i_N}$ is defined and closed on it.

The obvious "infinities" are directions when $(l_{\sigma(p+1)} - l_{\sigma(p)}) \to +\infty$ where $\exp(-(l_{\sigma(p+1)} - - l_{\sigma(p)})H) \to \Pi_W$.

The component of the differential form $\omega_{i_1\ldots i_N}$ that does contains neither $d\varphi_{\sigma(p+1)} - d\varphi_{\sigma(p)}$ nor $dl_{\sigma(p+1)} - dl_{\sigma(p)}$, tends to its factorized value, while components that contain any of them rapidly tend to zero, because $G_+\Pi_W = G_-\Pi_W = 0$.

The proper compactification of $(C^*)^{\otimes n}/C^*$ was introduced and studied in [17, 18]. In particular, it was shown that the set of forms $\omega_{i_1,\ldots,i_n}$ are maps to cohomologies of the new moduli spaces $\overline{L}_n$ taking values in linear operators in W that satisfy factorization property. It was shown that (similar to $\overline{M}_{0,n}$ case) all cycles in $\overline{L}_n$ could be represented as a sequence of spheres containing marked points and special points 0 and ∞, such the ∞ of the k-th sphere is attached to the 0 of the $k+1$ sphere. Below we will denote sphere with marked points by parenthesis containing labels of these points, and the sequence of spheres - by sequence of such parenthesises.

Relations between cycles in homology group of $\overline{L}_n$-spaces were studied in [18], it was shown that all these relations are generated as pullbacks of the Commutativity relation

$$(1)(2) = (2)(1) \qquad (15)$$

under the forgetfull map $\overline{L}_n \to \overline{L}_2$ that forgets positions of $(n-2)$ points.

From the factorization property of ω discussed above we see that if the sequence S_1S_2 is a junction of two sequences S_1 and S_2, then

$$\int_{S_1S_2} \omega_{i_1\ldots i_{m+n}} = \int_{S_1} \omega_{i_1\ldots i_m} \int_{S_2} \omega_{i_{m+1}\ldots i_{m+n}}. \qquad (16)$$

Evaluation of ω on cycles that are relations in cohomology gives equations for

$$\tau_{i_1\ldots i_m} = \int_{\overline{L}_m} \omega_{i_1\ldots i_m}.$$

If we consider the generating function

$$\tau(t) = \sum_m \frac{1}{m!}\tau_{i_1\ldots i_m} t^{i_1}\ldots t^{i_m} \qquad (17)$$

these equations would be equivalent to the Commutativity equations (known also as a t-part of $t - t^*$ equations [19]:

$$\frac{\partial \tau}{\partial t^i}\frac{\partial \tau}{\partial t^j} = \frac{\partial \tau}{\partial t^j}\frac{\partial \tau}{\partial t^i}. \qquad (18)$$

Now consider the product of two theories:

$$\mathcal{H}^{\mathrm{I\cdot II}} = \mathcal{H}^{\mathrm{I}} \otimes \mathcal{H}^{\mathrm{II}},\ Q^{\mathrm{I\cdot II}} = Q^{\mathrm{I}} + Q^{\mathrm{II}},\ G^{\mathrm{I\cdot II}} = G^{\mathrm{I}} + G^{\mathrm{II}}, \qquad (19)$$

$$\phi_I^{\mathrm{I\cdot II}} = \phi_i^{\mathrm{I}} \otimes \phi_{\bar{i}}^{\mathrm{II}}. \qquad (20)$$

The form $\omega^{\mathrm{I\cdot II}}$ on the space $\overline{L}_N$ is:

$$\omega^{\mathrm{I\cdot II}}_{I_1,\ldots,I_N} = \omega^{\mathrm{I}}_{i_1,\ldots,i_N} \otimes \omega^{\mathrm{II}}_{\bar{i}_1,\ldots,\bar{i}_N}. \qquad (21)$$

Thus,

$$\int_{\overline{L}_N} \omega^{\mathrm{I\cdot II}}_{I_1,\ldots,I_N} = (n^{-1})^{AB} \int_{C_A} \omega^{\mathrm{I}}_{i_1,\ldots,i_N} \otimes \int_{C_B} \omega^{\mathrm{II}}_{\bar{i}_1,\ldots,\bar{i}_N} \qquad (22)$$

here classes of C_A form a basis in the space of homologies of $\overline{L}_N$, n is the intersection matrix in homologies of $\overline{L}_N$.

Due to factorization property, $\int_{C_A} \omega_{i_1\ldots i_N}$ could be expressed in terms of products of $\int_{L_m} \omega_{i_1\ldots i_m}$, thus, $\tau^{\mathrm{I\cdot II}}$ could be expressed in terms of τ^{I} and τ^{II}, $\tau^{\mathrm{I\cdot II}} = \tau^{\mathrm{I}} \cdot \tau^{\mathrm{II}}$. Like in the case of product of solutions to WDVV equations the expression (22) could be considered as a formal product operation on solutons to Commutativity equations (even if they are not coming from any TST). In [20] it was proposed that starting from solution to Commutativity equation one can construct solution to Associativity equation as follows. We pick up a vector $h \in W$ and make a change of coordinates from t to θ^a:

$$\theta^a(t) = \tau^a_b(t)h^b. \qquad (23)$$

If matrix τ is symmetric with respect to some scalar product $\langle,\rangle$ and the change of coordinates is invertible, then

$$\tau_{ab} = \partial^2 F / \partial\theta^a \partial\theta^b \tag{24}$$

and F solves Associativity equation with the metric $\eta_{ab} = \delta_{ab}$.

We will denote the construction above as the CTA map from the pair τ, h (such that τ solves Commutativity equation) to $F(\theta)$ (such that F solves Associativity equations): $F = \mathrm{CTA}(\tau, h)$

Heuristically, vector h should be considered as a state that is generated by the functional integral over a disc in the theory with two-observable (corresponding to the first gravitational descendant) added to the action (in particular, it is well known [19] that adding to the action the first gravitational descendant of 1 corresponds to the rescaling of h).

The above heuristic considerations lead to the conjecture that can be rigorously checked and is really checked below in the first nontrivial case (corresponding to $\overline{M}_{0,5}$ and $\overline{L}_3$), that the CTA map is compatible with the product operation:

$$\mathrm{CTA}(\tau^{\mathrm{I}} \cdot \tau^{\mathrm{II}}, h^{\mathrm{I}} \otimes h^{\mathrm{II}}) = \mathrm{CTA}(\tau^{\mathrm{I}}, h^{\mathrm{I}}) \cdot \mathrm{CTA}(\tau^{\mathrm{II}}, h^{\mathrm{II}}). \tag{25}$$

It would extremely desirable to have a complete proof of the conjecture (25), but at a moment the direct check looks very complicated.

In preparation to the check of this conjecture we will make the linear parameter redefinition ($\theta \to T$):

$$\theta_a = T^i \frac{\partial \tau_{ab}}{\partial t^i}|_{t=0} h^b.$$

In these coordinates the metric equals to $\eta_{ij} = \langle h\tau_i\tau_j h\rangle$.

Let us get an explicit expression for the rhs of (25).

The five-point amplitude looks like:

$$F_{ijlkm} = \langle h\tau_i[\tau_{jkl}, \tau_m]h\rangle + \langle h[\tau_{im}, \tau_j]\tau_{kl}h\rangle +$$

$$\langle h[\tau_{im}, \tau_k]\tau_{jl}h\rangle + \langle h[\tau_{im}, \tau_l]\tau_{jk}h\rangle. \tag{26}$$

Let us choose the basis in the space of cycles such that the intersection matrix is nondegenerate(it has rank 5):

$$(ik)(jlm) + (jl)(ikm) +$$

$$+(jm)(ikl); (ij)(klm); (jk)(ilm); (jl)(ikm); (jm)(ikl). \tag{27}$$

In this basis the intersection matrix is diagonal: diag$(1; -1; -1; -1; -1)$

The five-point amplitude for the tensor product of two theories in terms of amplitudes of these theories can be symbolically rewritten as:

$$F^{1\otimes 2}_{(5)} = F_{(5)} \left(\overline{F}_{(3)}\overline{F}_{(3)}\overline{F}_{(3)}\right) + \left(F_{(3)}F_{(3)}F_{(3)}\right)\overline{F}_{(5)} +$$

$$+ \left(F_{(4)}F_{(3)}\right)_a \left(\overline{F}_{(3)}\overline{F}_{(4)}\right)_b (N^{-1})^{ab} \tag{28}$$

where $F_{(n)}$ denotes $F_{i_1\ldots i_n}$, the first two terms correspond to intersection of cycles of real dimension 4 and 0 and the last term corresponds to intersection of cycles of real dimension 2; $F_{(i)}; \overline{F}_{(i)}$ correspond to the first and second theory correspondingly. Using (28) we get

$$F^{1\otimes 2}_{IJKLM} = F_{ijklm}\langle \overline{h\tau_i\tau_j\tau_k\tau_l\tau_m h}\rangle + \langle h\tau_i\tau_j\tau_k\tau_l\tau_m h\rangle \overline{F}_{ijklm} -$$

$$-\langle h\tau_i\tau_j[\tau_{lm}, \tau_k]h\rangle\langle\overline{h\tau_i\tau_j[\tau_{lm}, \tau_k]h}\rangle -$$

$$-\langle h\tau_j\tau_k[\tau_{lm}, \tau_i]h\rangle\langle\overline{h\tau_j\tau_k[\tau_{lm}, \tau_i]h}\rangle -$$

$$-\langle h\tau_j\tau_l[\tau_{ik}, \tau_m]h\rangle\langle\overline{h\tau_j\tau_l[\tau_{ik}, \tau_m]h}\rangle -$$

$$-\langle h\tau_j\tau_m[\tau_{ik}, \tau_l]h\rangle\langle\overline{h\tau_j\tau_m[\tau_{ik}, \tau_l]h}\rangle +$$

$$+(\langle h\tau_i\tau_k[\tau_{lm}, \tau_j]h\rangle + \langle h\tau_j\tau_l[\tau_{ik}, \tau_m]h\rangle +$$

$$+\langle h\tau_j\tau_m[\tau_{ik}, \tau_l]h\rangle)(\langle\overline{h\tau_i\tau_k[\tau_{lm}, \tau_j]h}\rangle +$$

$$+\langle\overline{h\tau_j\tau_l[\tau_{ik}, \tau_m]h}\rangle + \langle\overline{h\tau_j\tau_m[\tau_{ik}, \tau_l]h}\rangle). \tag{29}$$

For $\overline{L}_3$ moduli space the intersection matrix of 2-cycles has rank 4. In the basis of cycles:

$$(jl)(k) + (j)(kl) + (jk)(l); (jk)(l); (jl)(k); (kl)(j) \tag{30}$$

the intersection matrix is diagonal: diag$(1; -1; -1; -1)$. The lhs of (25) can be written as:

$$F^{1\otimes 2}_{ijklm} = \langle h^{1\otimes 2}\tau_i^{1\otimes 2}[\tau_{jkl}^{1\otimes 2}, \tau_m^{1\otimes 2}]h^{1\otimes 2}\rangle +$$

$$+\langle h^{1\otimes 2}[\tau_{im}^{1\otimes 2}, \tau_j^{1\otimes 2}]\tau_{kl}^{1\otimes 2}h^{1\otimes 2}\rangle +$$

$$+\langle h^{1\otimes 2}[\tau_{im}^{1\otimes 2}, \tau_k^{1\otimes 2}]\tau_{jl}^{1\otimes 2}h^{1\otimes 2}\rangle +$$

$$+\langle h^{1\otimes 2}[\tau_{im}^{1\otimes 2}, \tau_l^{1\otimes 2}]\tau_{jk}^{1\otimes 2}h^{1\otimes 2}\rangle \tag{31}$$

where

$$h^{1\otimes 2} = h\otimes\overline{h}, \quad \tau_i^{1\otimes 2} = \tau_i\otimes\overline{\tau}_i,$$

$$\tau_{jk}^{1\otimes 2} = \tau_{jk}\otimes\overline{\tau_j\tau_k} + \tau_j\tau_k\otimes\overline{\tau_{jk}},$$

$$\tau_{jkl}^{1\otimes 2} = \tau_{jkl}\otimes\overline{\tau_j\tau_k\tau_l} + \tau_j\tau_k\tau_l\otimes\overline{\tau_{jkl}} +$$

$$+\tau_{kl}\tau_j\otimes\overline{\tau_{kl}\tau_j} + \tau_{kl}\tau_j\otimes\overline{\tau_l\tau_{jk}} +$$

$$+\tau_{kl}\tau_j\otimes\overline{\tau_k\tau_{jl}} + \tau_k\tau_{jl}\otimes\overline{\tau_{kl}\tau_j} +$$

$$+\tau_k\tau_{jl}\otimes\overline{\tau_l\tau_{jk}} + \tau_l\tau_{jk}\otimes\overline{\tau_{kl}\tau_j} +$$

$$+\tau_l\tau_{jk}\otimes\overline{\tau_k\tau_{jl}}-\tau_j\tau_{kl}\otimes\overline{\tau_j\tau_{kl}}. \qquad (32)$$

Substituting (32) into (31) one gets:

$$F^{1\otimes 2}_{ijklm}=\langle h\tau_j[\tau_{ilm},\tau_k]h\rangle\langle\overline{h\tau_i\tau_j\tau_k\tau_l\tau_m h}\rangle+$$
$$+\langle h\tau_i\tau_j\tau_k\tau_l\tau_m h\rangle\langle\overline{h\tau_j[\tau_{ilm},\tau_k]h}\rangle+$$
$$+\langle h\tau_i\tau_j\tau_k\tau_{lm} h\rangle\langle\overline{h\tau_i\tau_j\tau_k\tau_{lm} h}\rangle-$$
$$-\langle h\tau_i\tau_j\tau_{lm}\tau_k h\rangle\langle\overline{h\tau_i\tau_j\tau_{lm}\tau_k h}\rangle+$$
$$+\langle h\tau_j\tau_k\tau_m\tau_{il} h\rangle\langle\overline{h\tau_j\tau_k\tau_m\tau_{il} h}\rangle-$$
$$-\langle h\tau_j\tau_m\tau_{il}\tau_k h\rangle\langle\overline{h\tau_j\tau_m\tau_{il}\tau_k h}\rangle+$$
$$+\langle h\tau_j\tau_k\tau_l\tau_{im} h\rangle\langle\overline{h\tau_j\tau_k\tau_l\tau_{im} h}\rangle-$$
$$-\langle h\tau_j\tau_l\tau_{im}\tau_k h\rangle\langle\overline{h\tau_j\tau_l\tau_{im}\tau_k h}\rangle+$$
$$+(\langle h\tau_j\tau_{lm}\tau_i\tau_k h\rangle+\langle h\tau_j\tau_l\tau_{im}\tau_k h\rangle+\langle h\tau_j\tau_m\tau_{il}\tau_k h\rangle)\times$$
$$\times(\overline{\langle h\tau_j\tau_{lm}\tau_i\tau_k h\rangle+\langle h\tau_j\tau_l\tau_{im}\tau_k h\rangle+\langle h\tau_j\tau_m\tau_{il}\tau_k h\rangle})-$$
$$-(\langle h\tau_j\tau_k\tau_{lm}\tau_i h\rangle+\langle h\tau_j\tau_k\tau_l\tau_{im} h\rangle+\langle h\tau_j\tau_k\tau_m\tau_{il} h\rangle)\times$$
$$\times(\overline{\langle h\tau_j\tau_k\tau_{lm}\tau_i h\rangle+\langle h\tau_j\tau_k\tau_l\tau_{im} h\rangle+\langle h\tau_j\tau_k\tau_m\tau_{il} h\rangle})+$$
$$+\langle h[\tau_{jk},\tau_i]\tau_{lm} h\rangle\langle\overline{h\tau_i\tau_j\tau_k\tau_l\tau_m h}\rangle+$$
$$+\langle h\tau_i\tau_j\tau_k\tau_l\tau_m h\rangle\langle\overline{h[\tau_{jk},\tau_i]\tau_{lm} h}\rangle+$$
$$+\langle h\tau_i\tau_j\tau_k\tau_{lm} h\rangle\langle\overline{h[\tau_{jk},\tau_i]\tau_l\tau_m h}\rangle+$$
$$+\langle h[\tau_{jk},\tau_i]\tau_l\tau_m h\rangle\langle\overline{h\tau_i\tau_j\tau_k\tau_{lm} h}\rangle+$$
$$+\langle h[\tau_{jk},\tau_l]\tau_{im} h\rangle\langle\overline{h\tau_i\tau_j\tau_k\tau_l\tau_m h}\rangle+$$
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$$+\langle h\tau_j\tau_k\tau_l\tau_{im} h\rangle\langle\overline{h[\tau_{jk},\tau_l]\tau_i\tau_m h}\rangle+$$
$$+\langle h[\tau_{jk},\tau_l]\tau_i\tau_m h\rangle\langle\overline{h\tau_j\tau_k\tau_l\tau_{im} h}\rangle+$$
$$+\langle h[\tau_{jk},\tau_m]\tau_{il} h\rangle\langle\overline{h\tau_i\tau_j\tau_k\tau_l\tau_m h}\rangle+$$
$$+\langle h\tau_i\tau_j\tau_k\tau_l\tau_m h\rangle\langle\overline{h[\tau_{jk},\tau_m]\tau_{il} h}\rangle+$$
$$+\langle h\tau_j\tau_k\tau_m\tau_{il} h\rangle\langle\overline{h[\tau_{jk},\tau_m]\tau_i\tau_l h}\rangle+$$
$$+\langle h[\tau_{jk},\tau_m]\tau_i\tau_l h\rangle\langle\overline{h\tau_j\tau_k\tau_m\tau_{il} h}\rangle. \qquad (33)$$

Straightforward but tedious calculations show that (33) is Eq.(29), so (25) is fullfiled.

In this paper we motivated and checked the compatibility between product operations on solutions to Commutativity and Associativity equations in the first nontrivial case. It is not excluded that in more complicated cases these two product would give different results. If this happens, it would give a nontrivial indicator of whether solution to Commutativity equations came from TST. At the same time, if conjecture turns out to be a theorem, it would drastically simplify the calculation of the product on solutions to Associativity equations (because spaces $\overline{L}_n$ are much more tractable than $\overline{M}_{0,n}$).

A. L. is grateful to A. Gerasimov and I. P. is grateful to A. M. Levin for useful discussions. The work of A. L. is partially supported by grants: INTAS # 99-590, RFBR # 98-01-00328, # 00-15-96557 The work of I. P. is partially supported by grants: INTAS # 97-0103, RFBR # 98-01-00344 and grant for supporting of scientific schools RFBR # 96-15-96821.

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