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Nonlinear bloch waves

V. N. Serkin[1)], *M. Matsumoto*[+], *T. L. Belyaeva*[*]

Institute of General Physics RAS, 117942 Moscow, Russia

Benemerita Universidad Autonoma de Puebla, 72001 Puebla, Mexico

[+] *Osaka University, Osaka 565-0871, Japan*

[*] *Lomonosov Moscow State University, 119899 Moscow, Russia*

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The nonlinear Bloch theorem for the temporal and spatial Schrödinger solitons in dispersive and nonlinear periodic structures has been proven. It is shown that bright and dark solitary nonlinear Bloch waves exist only under certain conditions and that the parameter functions describing dispersion and nonlinearity periodic inhomogeneities cannot be chosen independently.

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A fundamental theorem concerning electrons in a crystal was proven by F. Bloch in 1928 [1]. It was shown that the wave functions of electrons in a periodic crystal lattice have the Bloch form

$$\Psi_k(x) = U_k(x)\exp(ikx), \tag{1}$$

where the function $U_k(x) = U_k(x+L)$ possesses the same periodicity as the lattice, and the wave number k is related to the de Broglie wavelength of the electron. The Bloch function in one-dimensional form (1) corresponds to a free electron wave function, $\exp(ikx)$, modulated by a function $U_k(x)$, which has the periodicity of the crystal lattice. The Bloch theorem is one of the basic concepts in solid state physics (see, for example, [2 – 6]).

In this Letter we show that there exists a nonlinear Bloch theorem for temporal and spatial Schrödinger solitons propagating through an inhomogeneous nonlinear and dispersive structures which are characterized by translational symmetry.

Let us consider the nonlinear Schrödinger equation model (NLSE) with periodically varying nonlinearity and dispersion:

$$i\frac{\partial\Psi^{\pm}}{\partial Z} \pm \frac{1}{2}D(Z)\frac{\partial^2\Psi^{\pm}}{\partial X^2} + R(Z)\mid\Psi^{\pm}\mid^2\Psi^{\pm} = 0. \tag{2}$$

Eq. (2) is written here in standard soliton units, as they are commonly known. It is assumed that periodic perturbations to the dispersion $D(Z) = D(Z+nL)$ and nonlinearity $R(Z) = R(Z+nL)$ are not limited to the regime where they are smooth and small. Here we also assume that the periods of those functions are arbitrary values. Due to a spatial-temporal analogy, both temporal and spatial solitons are described by Eq. (2).

Nonlinear Bloch theorem. The transformation law of the bright $\Psi^+(Z,X)$ and dark $\Psi^-(Z,X)$ NLSE solitons propagating through an inhomogeneous medium, which is characterized by translational symmetry $D(Z) = D(Z+nL)$ and/or $R(Z) = R(Z+nL)$, is defined by the self-reproducing stable configuration:

$$\Psi^{\pm}(Z) = \sqrt{P(Z)}\left\{\begin{array}{c}\sqrt{C}\eta\,\mathrm{sech}\,(\eta P(Z)X)\\ \sqrt{C}\eta\ \mathrm{th}\,(\eta P(Z)X)\end{array}\right\}\times$$

$$\times\exp\left[\pm i\frac{P(Z)}{2}X^2 + i\int\limits_0^Z K^{\pm}(\zeta)d\zeta\right], \tag{3}$$

where the real function $P(Z)$ possesses the same periodicity as the medium: $P(Z) = P(Z+nL)$. Nonlinear Bloch waves exist only under certain conditions given by the following relations:

$$P(Z) = P_0\left[1 + P_0\int\limits_0^Z \Phi(Z')dZ'\right]^{-1};$$

$$R(Z) = P(Z)\Phi(Z)/C, \tag{4}$$

where the dispersion parameter function is assumed to be given in the form of a periodical function $D(Z+nL) = \Phi(Z+nL)$.

[1)] e-mail:vserkin@hotmail.com

In another case, when the nonlinearity is assumed to be a given periodic function $R(Z) = R(Z+nL)$, the nonlinear Bloch wave exists only under conditions:

$$P(Z) = P_0 \exp\left[-C\int_0^Z R(Z')dZ'\right];$$

$$D(Z) = CR(Z)/P(Z). \qquad (5)$$

Parameter $C = D_0P_0/R_0$ in Eqs.(4), (5) is determined by the initial conditions at $Z = 0$.

Proof. Substitution of the nonlinear Bloch's function

$$\Psi^{\pm}(Z) = \sqrt{P(Z)}Q^{\pm}(S)\exp\left[\pm i\frac{P(Z)}{2}X^2 + i\int_0^Z K^{\pm}(\zeta)d\zeta\right], \qquad (6)$$

in (2) leads to

$$\pm\frac{1}{2}\frac{\partial^2 Q^{\pm}}{\partial S^2} + \frac{R}{DP}(Q^{\pm})^3 - Q^{\pm}\frac{K^{\pm}}{DP^2} \mp \frac{S^2Q^{\pm}}{2DP^4}\left(DP^2 + \frac{\partial P}{\partial Z}\right) = 0, \qquad (7)$$

$$\left(DP^2 + \frac{\partial P}{\partial Z}\right)\left(\frac{1}{2}Q^{\pm} + S\frac{\partial Q^{\pm}}{\partial S}\right) = 0. \qquad (8)$$

One can transform the general equation system (7), (8) into an exactly integrable form. Let us consider the complete nonlinear regime, when Eq.(7) represents the "classical", exactly integrable NLSE model and when two Eqs.(8) have a non-singular solution for Q-function. The necessary transformation can be expressed by the following equation system:

$$P^{-1}(Z) = C + \int D(\zeta)d\zeta;\ D(Z)P(Z) = R(Z)C. \qquad (9)$$

The eigenvalues $E^{\pm} = K^{\pm}/DP^2$ in (7) must satisfy $E^+ = 0.5\eta^2$ and $E^- = \eta^2$ for the bright and dark non-singular soliton solutions for Eqs.(7), (8):

$$Q(S)^{\pm} = \left\{\begin{array}{c} \eta\ \mathrm{sech}\ (\eta P(Z)X) \\ \eta\ \mathrm{th}\ (\eta P(Z)X) \end{array}\right\}. \qquad (10)$$

So, it is not surprising to find from Eqs.(9) that the soliton shape variation is also periodic $P(Z) = P(Z+nL)$, if the following relations are satisfied:

$$\frac{R(Z+nL)}{R(Z)} = \frac{D(Z+nL)}{D(Z)}. \qquad (11)$$

Notice that a necessary condition for the existance of a Bloch's theorem for the NLSE model with periodic gain or loss $\Gamma(Z) = \Gamma(Z+nL)$ is:

$$D(Z)P(Z) = R(Z)C\exp\left(2\int_0^Z \Gamma(\xi)d\xi\right). \qquad (12)$$

It should also be noted that exact integrability of the model (2) under conditions (9) means that there exists a transformation law (nonlinear Bloch's theorem) for the high-order soliton solutions of (2) as well.

Two features of the exact self-reproducing stable solutions (3)–(5), (9) are noteworthy.

1) Solutions (3)–(5), (9) can be considered as nonlinear Bloch waves with a periodic scattering potential which is reproduced by a solitary wave itself from cell to cell of a periodic structure. Unlike the homogeneous medium solution ($D(Z) \equiv 1$ and $R(Z) \equiv 1$), the soliton amplitude, pulse width and chirp (for the case of temporal solitons) or the radius of curvature of the wavefront (for the case of spatial solitons) have periodicity of the medium symmetry.

2) The dependence of soliton pulse width and phase profile on the propagation distance is defined by the same periodic function $P(Z)$. This remarkable result opens up the possibility to construct different nonlinear Bloch functions by using the algorithm proposed.

The fundamental set of nonlinear Bloch waves can be represented by Jacobi elliptic functions:

$$P(Z) = A + B\ dn^2(Z;s);$$

$$D(Z) = \frac{s^2B\ sn(2Z;s)\ \left[1 - s^2\ sn^4(Z;s)\right]}{\left[A + B\ dn^2(Z;s)\right]^2} \qquad (13)$$

where $dn^2 = 1 - s^2sn^2(Z;s)$, and $sn(Z;s)$ is the elliptical sinus function with the module parameter $m = s^2$. Periodic function $P(Z)$ represented by Eq.(13) has a period equal to $2K(s)$, where $K(s)$ is complete elliptic integral of the first kind [6]. Periodic function $P(Z)$ (13) transforms to a sinus-like wave for parameter $s \ll 1$:

$$P(Z) = A + B(1 - s^2\sin^2 Z);$$

$$D(Z) = s^2B\sin(2Z) \qquad (14)$$

(with asymptotic period given by $2K(s) = \pi(1 - s^2/4)$ for $s \ll 1$); and to a periodic train of sech-like solitons for $s \to 1$:

$$P(Z) = A + B\ \mathrm{sech}^2(Z);$$

$$D(Z) = \frac{2s^2 B\ \mathrm{sech}^2(Z)\ \mathrm{th}(Z)}{[A + B\ \mathrm{sech}^2\ (Z)]^2},$$

with period given by asymptotic value $2K(s) = \ln(16/(1-s^2))$ for $s \to 1$.

In the case, when a periodic structure is closed in the form of a loop with a total length equal to $Z = nL$, such a structure can be considered as an example of the soliton memory [7]. One of the simplest periodic solutions for Eq.(2) in this case is given by a periodic function $P(Z)$:

$$P(Z) = \frac{\sqrt{1-\beta^2}}{1-\beta\cos kZ}, \qquad (15)$$

which, for $\beta \to 1$, becomes a periodic grating of delta-functions, and, for $\beta \to 0$, a constant $P(Z) \to 1$. Then, the nonlinear Bloch theorem (3) leads to the requirement that dispersion and nonlinearity of the soliton memory loop must be given by periodic functions:

$$D(Z) = \frac{\beta k \sin kZ}{\sqrt{1-\beta^2}};\ R(Z) = \frac{\beta k \sin kZ}{1-\beta\cos kZ}. \qquad (16)$$

It seems very attractive to use the nonlinear Bloch concept to design novel types of soliton lasers. The most noteworthy feature of the quasi-steady-state soliton laser operation scenario is the fact that the gain and losses are exactly compensated during one soliton pulse round trip. One can model the effective distributed gain and losses inside the laser cavity by a periodic function, such as

$$\Gamma(Z) = \frac{\sin Z}{\Delta^3};\ \text{where}\ \Delta = \sqrt{1-\delta^2\sin^2 Z}. \qquad (17)$$

The nonlinear Bloch theorem (3,4) states that the corresponding dispersion function must satisfy

$$D(Z) = -2\frac{\sin Z}{\Delta^3}\exp\left[-\frac{2}{(1-\delta^2)}\left(1-\frac{\cos Z}{\Delta}\right)\right]. \qquad (18)$$

The main soliton features of the nonlinear solitary Bloch waves predicted analytically have been investigated by using direct computer simulations. The nonlinear Bloch waves scenario for the case represented by Eqs.(3), (15), (16) is shown in Fig.1. Nontrivial (nonzero) initial velocities in the co-moving frame of reference are given via the Galileian transformation of (3). A periodic "snake" effect arises in the space-time domain for solitons with nonzero initial velocities in the retarded frame of reference (see Fig.2). In Fig.3 we illustrate typical interactions between nonlinear solitary Bloch waves. An important feature of the solitary nonlinear Bloch waves consists in an elastic character of their interaction, which is shown in Fig.3.

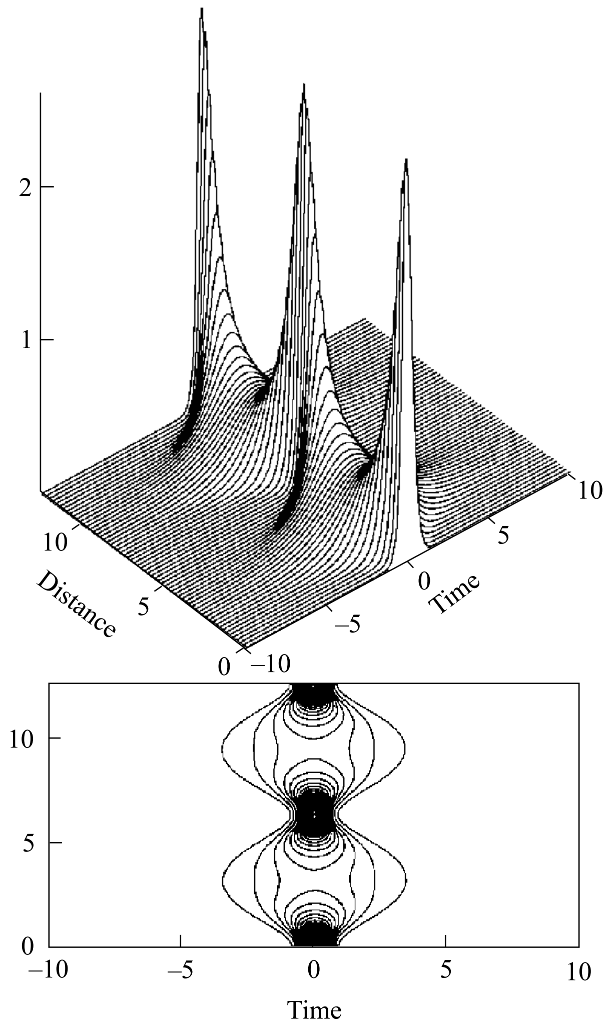

Fig.1. Evolution of the nonlinear solitary Bloch wave (Eqs.(3) and (15), (16)) as a function of the propagation distance. Initial conditions: $k = 1$; $\beta = 0.5$

Recently, the possibility of spatial self-trapping of quasi periodic waves due to cascaded self-focusing was numerically demonstrated, and the quasi periodic envelope soliton concept was introduced [8]. The concept of nonlinear Bloch waves, which are localized nonlinear waves in a periodic structures, was proposed for the first time by Haus and Chen [9]. Haus and Chen tried to construct the steady-state solutions for the nonlinear problem as a superposition of the Hermit-Gaussians polynomials of the linear propagation problem [9]. In this Letter the exact solution of the nonlinear problem is obtained. It is shown that solitary nonlinear Bloch waves exist only under certain conditions, and the nonlinear Bloch theorem is proven.

In summary, we predict a novel class of nonlinear solitary bright and dark Bloch waves in inhomogeneous media which are characterized by translational symmetry. "Classical" soliton-like features of the nonlinear Bloch waves are confirmed by accurate direct computer simulations. Nonlinear Bloch's theorem obtained in this Letter is of general physics interest and should be readily verified experimentally in periodic nonlinear and dispersive structures in different branches of physics, where the "universal" NLSE model is applicable. For example, it seems very attractive to use the nonlinear Bloch wave concept in ultrashort pulse photonics applications and soliton lasers design [10]. The best soliton laser performance is obtained,

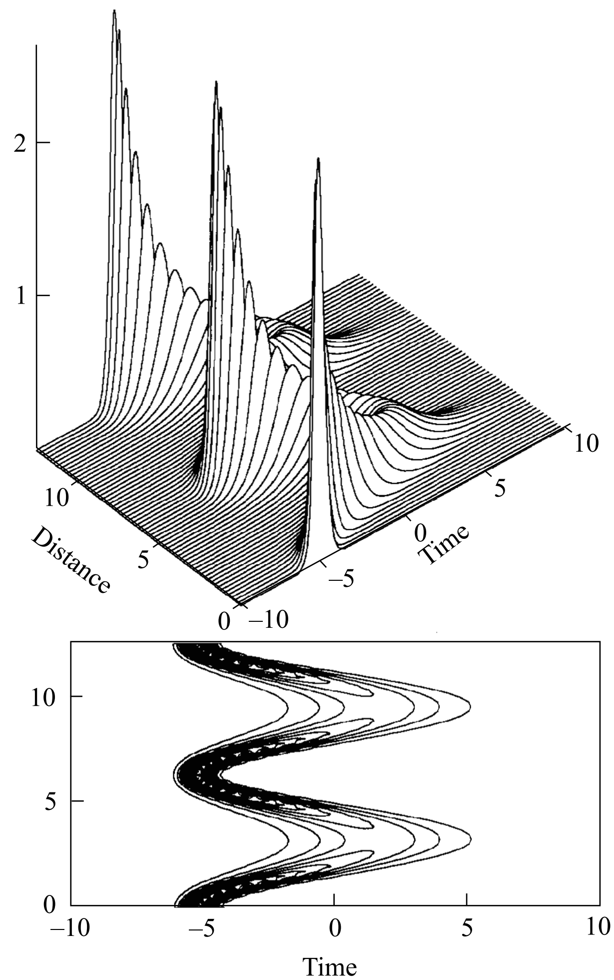

Fig.2. Evolution of the nonlinear solitary Bloch wave (Eqs.(3) and (15), (16)) with nontrivial initial conditions as a function of the propagation distance: soliton "snake" effect. Initial conditions: initial group velocity $V = 10$; $k = 1$; $\beta = 0.5$

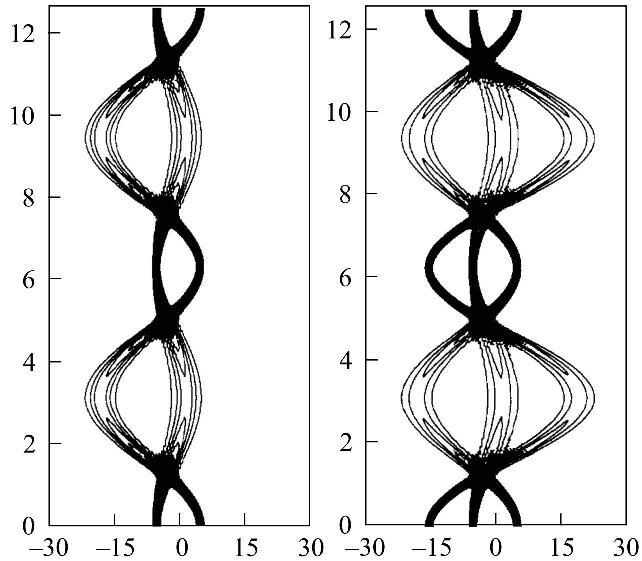

Fig.3. Elastic interaction between two and three solitary nonlinear Bloch waves in periodic structure given by Eqs.(15), (16). In the case of temporal solitons, X-axis corresponds to the dimensionless time in retarded frame; in the case of two-dimensional spatial solitons X-axis corresponds to the transverse coordinate. Y-axis represents the normalized distance of the nonlinear Bloch waves propagation

when there is a sign-reversal periodic dispersion and/or nonlinearity inside the laser cavity, according to nonlinear Bloch theorem.

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