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Kolmogorov spectra in one-dimensional weak turbulence

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In this article we report results of our numerical simulation of one-dimensional modified MMT-model, which includes processes "one to three" wave interactions. We show that this model, at a proper choice of parameters, behaves accordingly to the weak-turbulent theory. In particular, it demonstrates validity of the Kolmogorov spectrum in the wide range of wavenumbers.

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1. The problem of Kolmogorov spectra is the core of the theory of weak wave turbulence. These spectra appear as exact solutions of stationary kinetic equation for mean squared wave amplitudes [1]. On our opinion, the weak turbulent Kolmogorov spectra should be used for theoretical explanation of powerlike spectral distributions of energy in ensembles of stochastic nonlinearly interacting waves of any nature. Spectra of such type are observed systematically. The most conspicuous example of that sort is the spectrum $\epsilon_\omega \simeq gv/\omega^4$, which is routinely observed in the systems of wind-driven gravity sea waves.

However, this view-point is not shared by everybody. Moreover, the very applicability of kinetic equation for wave to real situation is discussed (see, for instance [2]). Derivation of the kinetic equation from initial dynamic equation presumes validity of assumption of phase randomness, which can be destroyed by formation of some coherent structures, like solitons or wave collapses. Actually, this critic has serious foundations. In real situations the coherent structures are common, but there is no reason for complete abandoning of the weak turbulent theory. The real life is multicolour, and in many particular situations the coherent structures coexist with weak turbulence, sharing processes of transport and dissipation of energy and other integrals of motion.

Hence, there is a strong motivation to continue the study of weak turbulence, exploring both the case where the coherent structures are important as well as the case when the influence of such structures is negligible.

2. One of the most promising approaches for the study of weak turbulence is a direct numerical simulation of nonlinear dynamic equations, describing wave systems. In many cases these equations can be effectively solved by the use of a spectral code. Of course, the numerical simulation of the equation completely relevant to real physical situation is the most preferable. However, a lot of interesting information can be extracted from the solution of simplified, more or less artificial models, which inherit the basic features of the real physical equations. As far as the weak turbulence is a very general theory, its main statements – applicability of the kinetic equation, existence of Kolmogorov-type spectra and structures of high-order correlation functions, etc., – can be properly tested on these simple models, for which the computer simulation can be done more easily.

The one-dimensional models are the most attractive ones from this point of view. Even a modest contemporary computer makes possible to perform a numerical simulation of a system of nonlinear waves including thousand modes and three decades of scaling. Historically the first such simulation was done in 1997 [2]. They used the model which is called now after their names as MMT-model. The MMT-model is a generalized nonlinear Schrödinger equation, preserving not only energy and momentum, but the wave action (number of particles) as well.

Further simulation of the MMT-model was done by the same authors in collaboration [3]. Later on numerical experiments on the MMT-model were performed [4, 5]. The results of both groups basically coincide. The MMT-model demonstrates a complicated many-variant behavior which cannot be considered as a certain confirmation of the weak-turbulent theory. On our opinion, this is because of an interference of the coherent structures which are persistent in all versions of MMT-model.

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In this article we report results of our numerical simulation of a modified MMT-model, which includes processes "one to three" wave interactions, not preserving wave action. We will show that this model, at a proper choice of parameters, behaves accordingly to the weak-turbulent theory. In particular, it demonstrates validity of the Kolmogorov spectrum in the range of more than two decades.

3. We study the following model:

$$i\left(\frac{\partial\psi}{\partial t}+\gamma_k\psi_k\right)=k^{\alpha}\psi+$$
$$+a\left\{\int\left(|k||k_1||k_2||k_3|\right)^{\beta/4}\times\right.$$
$$\times\psi^*_{k_1}\psi_{k_2}\psi_{k_3}\delta_{k+k_1-k_2-k_3}dk_1dk_2dk_3+$$
$$+g\int\left(|k||k_1||k_2||k_3|\right)^{\beta/4}\left(\psi_{k_1}\psi_{k_2}\psi_{k_3}\delta_{k-k_1-k_2-k_3}+\right.$$
$$\left.\left.+3\psi_{k_1}\psi^*_{k_2}\psi^*_{k_3}\delta_{k-k_1+k_2+k_3}\right)dk_1dk_2dk_3\right\}. \qquad (1)$$

If $g=0$, this model becomes the MMT-model.

The model (1) can be written as

$$i\left(\frac{\partial\psi}{\partial t}+\gamma_k\psi_k\right)=\frac{\delta H}{\delta\psi^*_k}, \qquad (2)$$

where

$$H=H_0+H_{int},$$
$$H_0=\int|k|^{\alpha}\psi_k\psi^*_kdk, \qquad (3)$$

$$H_{int}=a\left\{\frac{1}{2}\int(kk_1k_2k_3)^{\beta/4}\psi^*_k\psi^*_{k_1}\psi_{k_2}\psi_{k_3}\delta_{k+k_1-k_2-k_3}+\right.$$
$$+g\int(kk_1k_2k_3)^{\beta/4}\left[\psi^*_k\psi^*_{k_1}\psi^*_{k_2}\psi_{k_3}+\psi_k\psi_{k_1}\psi_{k_2}\psi^*_{k_3}\right]\times$$
$$\left.\times\delta_{k+k_1+k_2-k_3}dk_1dk_2dk_3\right\}. \qquad (4)$$

Hamiltonian H describes the following four-wave processes:

a) Scattering, obeying the resonant conditions

$$\omega_k+\omega_{k_1}=\omega_{k_2}+\omega_{k_3},\quad k+k_1=k_2+k_3. \qquad (5)$$

b) Decay "one wave to three" and the reverse process of gluing of three waves to one wave, obeying the resonant conditions

$$\omega_k=\omega_{k_1}+\omega_{k_2}+\omega_{k_3},\quad k=k_1+k_2+k_3. \qquad (6)$$

Here $\omega_k=|k|^{\alpha}$.

We have studied only the case $\alpha>1$. In this case resonant conditions (5) have only a trivial solution

$$k_2=k,\ k_3=k_1 \text{ or } k_2=k_1,\ k_3=k, \qquad (7)$$

while resonant conditions (6) describe $2-D$ manifold in space (k,k_1,k_2,k_3). If $a>0$ and g is small, Hamiltonian (4) is positively defined. It makes possible to get rid off any kinds of localized structures.

Under these assumptions system (1) is described by the kinetic equation

$$\partial n/\partial t+2\gamma_k n_k+st(n,n,n), \qquad (8)$$

where

$$st(n,n,n)=4\pi a^2g^2\left\{\int(kk_1k_2k_3)^{\beta/2}\times\right.$$
$$\times(n_{k_1}n_{k_2}n_{k_3}-n_kn_{k_1}n_{k_2}-n_kn_{k_1}n_{k_3}-n_kn_{k_2}n_{k_k})\times$$
$$\times\delta(k-k_1-k_2-k_3)\delta(\omega_k-\omega_{k_1}-\omega_{k_2}-\omega_{k_3})dk_1dk_2dk_3+$$
$$+3\int(kk_1k_2k_3)^{\beta/2}(n_{k_1}n_{k_2}n_{k_3}+n_kn_{k_1}n_{k_2}+$$
$$+n_kn_{k_1}n_{k_3}-n_kn_{k_2}n_{k_3})\delta(k-k_1+k_2+k_3)\times$$
$$\left.\times\delta(\omega_k-\omega_{k_1}+\omega_{k_2}+\omega_{k_3})dk_1dk_2dk_3\right\}. \qquad (9)$$

By definition

$$\langle\psi_k\psi^*_{k'}\rangle=n_k\delta_{k-k'}. \qquad (10)$$

A stationary equation

$$st(n,n,n)=0 \qquad (11)$$

has, for a proper values of α and β, a powerlike solution

$$n_k=\alpha p^{1/3}/k^{\lambda}, \qquad (12)$$

$$\lambda=\frac{2}{3}\beta+1. \qquad (13)$$

This is the Kolmogorov spectrum carrying to large k region a constant flux of energy p, and α is a dimensionless Kolmogorov constant.

4. We have performed a numerical simulation of equation (1) by the use of a standard spectral code. We put $\alpha=3/2,\beta=9/4,a=1$ at different values of the dimensionless parameter $g=0,0.05,0.1,0.15,0.2$.

Our spectral array included 2048 modes, $-1024<k<1023$. The system was pumped at low wave numbers, $\gamma=-0.005$ at $5\leq|k|\leq10$. The sink of energy at large wave numbers was provided by damping, $\gamma_n=400(k/512-0.5)^2$ at $|k|>512$.

In all variants of our computations we have been observing growth and stabilization of total energy of wave system H. According to weak turbulent

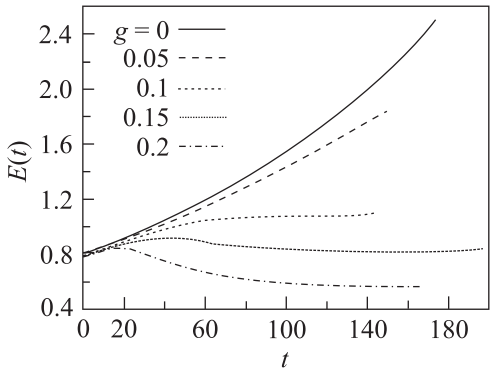

Fig.1. Total energy of the pumped system versus time for different coefficients of "tree to one" process

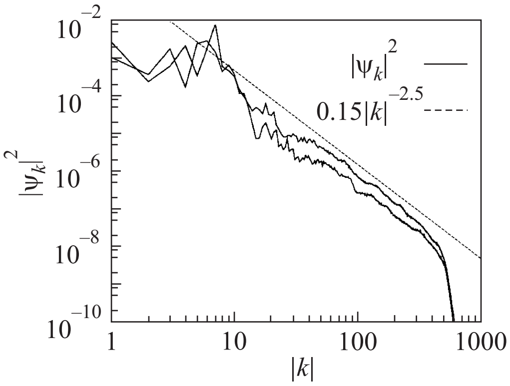

Fig.2. $|\psi_k|^2$ averaged over time 100. Spectra for positive and negative k are shown

theory stabilization level depends dramatically on the parameter g.

Fig.1 clearly demonstrates that $1 \Longleftrightarrow 3$ processes play the main role in establishing of equilibrium.

Fig.2 displays typical stationary spectra at $g = 0.15$. One can see that in a range $30 < k < 300$ they can be well approximated by the Kolmogorov exponent $\lambda = 5/2$. A typical value of nonlinearity,

$$\epsilon = H_{\rm int}/H,$$

is $\epsilon \simeq 0.15$.

In conclusion we would like to claim that our result is the first clear confirmation of validity of the weak turbulent theory in $1-D$ case. A similar confirmation for $2-D$ case was done in the work [6]. However in the presented $1-D$ case the range of scales where Kolmogorov spectrum is observed is essentially larger.

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