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Quantum limits of feedback cooling in optical lattices

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A quantum mechanical analysis of feedback cooling [see Phys. Rev. Lett. **94**, 153002 (2005)] of atoms trapped in a far-off-resonant optical lattice is presented. The model considered is valid for ultralow energies of atoms allowing thus for the study of an ultimate cooling limit. The influence of the measurement-induced noise and feedback-induced atom-atom correlations on the cooling efficiency is discussed. It is shown that there are regimes where the quantum noise can be effectively compensated for.

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At present manipulation and control of single atoms and atomic ensembles represent important challenges for experimentalists and theoreticians. A widely used strategy for manipulating atoms is based on the resonant interaction of atoms with laser fields [1]. However, application of resonant fields considerably restricts the controllability due to the noise induced by spontaneous emission. Moreover, only atomic species possessing specific cyclic transitions resonant with available lasers can be addressed in this case.

On the other hand, the range of accessible atomic species can be extended and the spontaneous-emission noise can be diminished if one would be able to use far-off-resonant laser fields to control atoms. In this case the mechanism of manipulation would be the nonresonant interaction of the induced atomic dipole moment with laser light (AC Stark effect) which results in a dipole force acting on atoms. This mechanism can be conveniently realized in far-off-resonant optical lattices. An optical lattice is a periodic light-shift potential seen by neutral atoms located inside an interference pattern formed by laser beams tuned far from the atomic transition. Adjusting wavelength, polarization, and intensity of the lasers forming the optical lattice one can flexibly control the motion of cold atoms [2 – 5].

In particular, some years ago the possibility of using feedback to control the motion of cold atoms in an one-dimensional optical lattice has been experimentally demonstrated [6]. As is also proposed in Ref. [6], the concept of feedback can be applied to cool atoms in a manner similar to stochastic cooling [7, 8]. In the recent paper [9] the realization of such a feedback cooling (termed as "optical shaking") has been elaborated in more detail and the operation of this cooling technique has been studied in the classical limit. The classical approach developed in [9] can be successfully applied to describe cooling in the case of sufficiently high energies of atoms. However, it does not apply in the limit of low energies when quantum effects start to play a significant role. In this case the noise introduced into the system due to the measurement can strongly restrict the cooling efficiency and therefore should not be ignored. Furthermore, as will be shown below the cooling efficiency is also influenced by atom-atom correlations, which appear due to the feedback process itself. The effect of these correlations is not evident from the discussions in Ref. [9].

In this letter the feedback cooling of atoms in optical lattices is studied on the basis of the quantum theory. Such an approach will allow us to study the effect of both feedback-induced noise and atom-atom correlations on the cooling efficiency at low temperatures and estimate the cooling limit as well.

We restrict our consideration to the case of one-dimensional optical lattice. In the limit of low temperatures the optical lattice potential can be approximated by the array of harmonic potential wells with each well containing only one atom. The tunneling of atoms between neighboring potential wells can be neglected, which allows us to distinguish atoms according to the specific potential well they belong to. Consequently, the quantum statistics of atoms is out of relevance in the considered model. Furthermore, being trapped in separate potential wells the atoms interact weakly with each other, which does not affect the evolution of the system considerably and can be also neglected. In this case as will be shown below a rather complete analysis of the evolution of the system's state is possible.

The feedback loop is organized as follows: The first step is the measurement of the collective coordinate of

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atoms, which is the position of atoms with respect to the center x_i of their wells averaged over all wells or, equivalently, over all atoms

$$\hat{Q} = \frac{1}{N}\sum_{i=1}^{N}(\hat{q}_i - x_i). \qquad (1)$$

Here N is a total number of atoms, which is assumed to be fixed and $\hat{q}_i$ is the coordinate of the i-th atom. The constant c-number

$$\frac{1}{N}\sum_{i=1}^{N} x_i \qquad (2)$$

can be ignored.

The average coordinate (1) is obtained experimentally measuring the imbalance in intensities of laser beams forming the lattice, which appears due to the interaction with atoms, see Ref. [6] for experimental details.

Then the measurement outcome Q of the observable $\hat{Q}$ is used to perform the spatial translation of the optical-lattice potential to compensate for the measured coordinate. The compensation of this coordinate leads to the decrease of the total potential energy of atoms and, hence, to cooling of the gas.

The time duration of the measurement and the lattice shift is assumed to be negligibly small compared with the characteristic time of the free evolution of atoms in the lattice potential [6]. Therefore the oscillations of atoms in the lattice potential during the feedback operation will be neglected.

The quantum state of the system after a single feedback step is given by the density operator $\hat{\rho}_+$:

$$\hat{\rho}_+ = \int dQ\, \hat{U}(Q)\hat{M}(Q)\,\hat{\rho}_-\,\hat{M}^+(Q)\hat{U}^+(Q). \qquad (3)$$

Here $\hat{\rho}_-$ is the many-atom density operator before the feedback, the operator $\hat{M}(Q)$ describes the effect of the measurement on the system conditioned on the measurement outcome Q, and $\hat{U}(Q)$ is the unitary shift of the collective coordinate $\hat{Q}$ to zero. In general, feedback is an automatic process requiring no intervention of an experimentalist. Thus the quantum state of the system after the feedback is obtained averaging over the all possible measurement outcomes Q.

To include imprecision measurements are described in terms of operators constituting a positive operator-valued measure (POVM) [10]. Thus the operators $\hat{M}(Q)$ appearing in Eq. (3) are the measurement resolution amplitudes constituting POVM. The simplest choice for these operators is a Gaussian operator

$$\hat{M}(Q) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(\hat{Q}-Q)^2}{4\sigma^2}\right], \qquad (4)$$

where σ is the resolution of a measurement device.

The shift of the collective coordinate of atoms to zero being actually a one-parametric shift transformation is described by the unitary operator

$$\hat{U}(Q) = \exp\left(iQ\hat{P}\right), \qquad (5)$$

where the generator

$$\hat{P} = \sum_{i=1}^{N}\hat{p}_i \qquad (6)$$

is the total momentum of the atoms. The operator $\hat{p}_i$ in this equation is the momentum of the i-th atom. The operator $\hat{P}$ is the canonical conjugate to $\hat{Q}$ with the standard commutator $[\hat{Q},\hat{P}] = i$ [2)]. It may be shown using this commutator that the operator $\hat{U}(Q)$ acts on the observable $\hat{Q}$ as

$$e^{-iQ\hat{P}}\hat{Q}e^{iQ\hat{P}} = \hat{Q} - iQ[\hat{P},\hat{Q}] = \hat{Q} - Q \qquad (7)$$

resulting in the required shift of $\hat{Q}$ by $-Q$.

It is advantageous to use a phase-space representation of the quantum state of the system introducing the many-atom Wigner function defined as

$$W(\mathbf{p},\mathbf{q}) = \frac{1}{\pi^N}\int d^N y\, e^{-2i\mathbf{y}\mathbf{p}}\langle \mathbf{q}+\mathbf{y}|\hat{\rho}|\mathbf{q}-\mathbf{y}\rangle. \qquad (8)$$

Here $\mathbf{q} = \{q_1, q_2, \ldots, q_N\}$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_N\}$ denote the vectors with components being the single-atom coordinates and momenta, respectively. The many-atom basis in Eq. (8) is the product of the single-atom basis vectors, so that

$$|\mathbf{q}-\mathbf{y}\rangle = |q_1 - y_1\rangle|q_2 - y_2\rangle\ldots|q_N - y_N\rangle. \qquad (9)$$

Using the definition (8) and Eq. (3) the Wigner function after a single feedback step can be written as

$$W_+^{(1)}(\mathbf{p},\mathbf{q}) = \pi^{-N}\int dQ\, d^N y\, du\, du'\, \mathcal{M}(u)\mathcal{M}^*(u') \times$$
$$\times \exp\left[-\frac{i(u-u')}{N}\sum_{i=1}^{N} q_i - \frac{i(u+u')}{N}\sum_{i=1}^{N} y_i\right] \times$$
$$\times \exp(-2i\mathbf{y}\mathbf{p})\,\langle \mathbf{q}+\mathbf{y}+Q\mathbf{e}|\,\hat{\rho}_-\,|\mathbf{q}-\mathbf{y}+Q\mathbf{e}\rangle, \qquad (10)$$

[2)] We use $\hbar = 1$ throughout the paper.

where $\mathbf{e}$ is the vector with N unit components: $\mathbf{e} = \{1, 1, \ldots, 1\}$. Here the Fourier representation of the measurement resolution amplitudes $\hat{M}(Q)$

$$\hat{M}(Q) = \int du\, \mathcal{M}(u) \exp\left[iv(Q - \hat{Q})\right], \qquad (11)$$

with c-functions $\mathcal{M}(u)$

$$\mathcal{M}(u) = \left(\frac{\sigma_1^2}{2\pi^3}\right)^{1/4} \exp(-\sigma_1^2 u^2) \qquad (12)$$

has been used. Changing now the variables $u+u' = v$ and $u-u' = v'$ in Eq. (10) and integrating then over v' the Wigner function after the feedback operation reads

$$W_+^{(1)}(\mathbf{p}, \mathbf{q}) = \frac{1}{2\pi} \exp\left(-\frac{1}{2\sigma_1^2 N^2} \sum_{i,j=1}^{N} q_i q_j\right) \times$$
$$\times \int dQ\, dv\, \exp\left(-\frac{\sigma_1^2 v^2}{2}\right) W_-\left(\mathbf{p}+\frac{v}{2N}\mathbf{e},\, \mathbf{q}+Q\mathbf{e}\right), \qquad (13)$$

where W_- denotes the Wigner function before the feedback. This result can be rewritten in a compact form introducing the following integral kernel

$$\mathcal{F}^{(1)}(\mathbf{p}, \mathbf{q}; \mathbf{p}', \mathbf{q}') = \frac{1}{2\pi} \exp\left(-\frac{1}{2\sigma_1^2 N^2} \sum_{i,j=1}^{N} q_i q_j\right) \times$$
$$\times \int dQ\, dv \exp\left(-\frac{\sigma_1^2 v^2}{2}\right) \times$$
$$\times\, \delta\left(\mathbf{p}'-\mathbf{p}-\frac{v}{2N}\mathbf{e}\right) \delta(\mathbf{q}'-\mathbf{q}-Q\mathbf{e}), \qquad (14)$$

which gives

$$W_+^{(1)}(\mathbf{p}, \mathbf{q}) = \int d^N p' d^N q'\, \mathcal{F}^{(1)}(\mathbf{p}, \mathbf{q}; \mathbf{p}', \mathbf{q}') W_-(\mathbf{p}', \mathbf{q}'). \qquad (15)$$

The integral transformation (15) with the kernel (14) represents the core component of the solution to the feedback problem and will be used below to study the dynamics of the system due to a number of feedback steps.

To do this the transformation of the Wigner function due to the free evolution of atoms between feedbacks should be included. Assuming that the time interval between two successive feedbacks is small enough, so that the noise does not affect the system considerably, the evolution of atoms between the feedbacks is governed by the free Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{p}_i^2}{2m} + \frac{m\omega^2}{2}\hat{q}_i^2\right), \qquad (16)$$

and results in a simple rotation of the Wigner function

$$W(\mathbf{p}, \mathbf{q}, t+\Delta t) = W(\mathbf{p}', \mathbf{q}', t),$$
$$\mathbf{p}' = m\omega\, \mathbf{q} \sin(\omega\Delta t) + \mathbf{p} \cos(\omega\Delta t), \qquad (17)$$
$$\mathbf{q}' = \mathbf{q} \cos(\omega\Delta t) - (1/m\omega)\mathbf{p} \sin(\omega\Delta t),$$

where Δt denotes the time interval after the last feedback step, m is the atomic mass and ω is the vibrational trap frequency. Now the resulting Wigner function after arbitrary number of feedbacks and arbitrary duration of free evolution between them can be found applying Eqs (15) and (17) in an appropriate sequence.

Aiming to effectively compensate the collective coordinate of atoms one can try to find an optimal time interval between feedbacks. This optimal interval can be fixed considering the collective motion of the atoms in the harmonic potential classically. It is easily seen that taking the time interval between feedback steps equal to the one fourth of the oscillation period $\Delta t = \tau/4$ the collective motion is completely damped after two feedback steps. Given another time of free evolution this would normally take more operations.

Using Eqs (15) and (17) with $\Delta t = \tau/4$, the Wigner function of the system immediately after the k-th feedback step ($k \geq 2$) is expressed as

$$W_+^{(k)}(\mathbf{p}, \mathbf{q}) = \int d\mathbf{p}' d\mathbf{q}'\, \mathcal{F}^{(k)}(\mathbf{p}, \mathbf{q}; \mathbf{p}', \mathbf{q}') W_-(\mathbf{p}', \mathbf{q}') \qquad (18)$$

with the feedback kernel given by

$$\mathcal{F}^{(k)}(\mathbf{p}, \mathbf{q}; \mathbf{p}', \mathbf{q}') = \frac{1}{2\pi}(4\sigma_k^4 N^2 + 1)^{-1/2} \times$$
$$\times \exp\left(-\frac{1}{2\sigma_k^2 N^2} \sum_{i,j=1}^{N} q_i q_j\right) \times$$
$$\times \exp\left(-\frac{2\sigma_k^2}{4\sigma_k^4 N^2 + 1} \sum_{i,j=1}^{N} p_i p_j\right) \times$$
$$\times \int dQ dv\, \delta\left[\mathbf{q}' + \cos\left(\pi\frac{k+1}{2}\right)\mathbf{q} - \sin\left(\pi\frac{k+1}{2}\right)\mathbf{p} + Q\mathbf{e}\right] \times$$
$$\times\, \delta\left[\mathbf{p}' + \sin\left(\pi\frac{k+1}{2}\right)\mathbf{q} + \cos\left(\pi\frac{k+1}{2}\right)\mathbf{p} - \frac{v}{2N}\mathbf{e}\right]. \qquad (19)$$

Given the measurement resolution being the same at each feedback step, because of the 2π-periodicity of the feedback kernel, Eq. (19), the state of the system will reach a quasi-stationary regime where the state after the second step will be revealed with the four-step periodicity. In the case of symmetric initial Wigner function the period of oscillations can be shorter. For example, given a thermal equilibrium state or a Fock state, the Wigner function after the second feedback step is not affected by the next feedback operations. This represents certain limitations of the cooling scheme.

To study the operation of feedback cooling in more detail let us now specify the initial state of the atoms. In most cases the initial state of the atoms can be represented by a thermal equilibrium state. Given distinguishable atoms in a thermal equilibrium state the atom-atom correlations are absent. Thus, the initial state of the many-atom system is then the tensor product of the states of single atoms $\hat{\rho}_- = \hat{\rho}_1^{\rm th} \otimes \hat{\rho}_2^{\rm th} \otimes \cdots \otimes \hat{\rho}_N^{\rm th}$. Consequently the Wigner function is the product of the thermal Wigner functions of N single atoms. It is given by

$$W_-(\mathbf{p},\mathbf{q}) = \\ = \left[\frac{\tanh(1/\xi)}{\pi}\right]^N \exp\Big[-\frac{\tanh(1/\xi)}{2}\sum_{i=1}^N (q_i^2+p_i^2)\Big], \quad (20)$$

where q_i and p_i are expressed in units of the ground-state position and momentum uncertainty in the trap potential $\Delta q_0 = \sqrt{1/(2m\omega)}$ and $\Delta p_0 = \sqrt{m\omega/2}$, respectively. The parameter $\xi = k_{\rm B}T/(\omega/2)$ denotes the single-atom energy in units of the ground state energy in the trap. $k_{\rm B}$ is the Boltzmann constant.

From Eq. (15) after the evaluation of corresponding integrals the Wigner function after a single feedback is found in the form

$$W_+^{(1)}(\mathbf{p},\mathbf{q}) = \\ = \left[\frac{\tanh(1/\xi)}{\pi}\right]^N \left[\frac{\coth(1/\xi)}{N\sigma_1^2 + \tanh(1/\xi)}\right]^{1/2} \times \\ \times \exp\left[-\frac{1+N\sigma_1^2(N-1)\tanh(1/\xi)}{2N^2\sigma_1^2}\sum_{i=1}^N q_i^2 - \right. \\ \left. - \frac{1-N\sigma_1^2\tanh(1/\xi)}{2N^2\sigma_1^2}\sum_{i\neq j=1}^N q_i q_j\right] \times \\ \times \exp\left[-\frac{(N-1)\tanh(1/\xi)+N^2\sigma_1^2}{2N[1+N\sigma_1^2\coth(1/\xi)]}\sum_{i=1}^N p_i^2 + \right. \\ \left. + \frac{\tanh(1/\xi)}{2N[1+N\sigma_1^2\coth(1/\xi)]}\sum_{i\neq j=1}^N p_i p_j\right], \quad (21)$$

where σ_1 is now also measured in units of Δq_0. From this equation one can readily see the appearance of correlations between coordinates (the second term in the first exponent) and momenta (the second term in the second exponent) of different atoms. The factor in front of the double sum over coordinates can be either negative or positive depending on the values of the number of atoms, the measurement resolution, and the parameter ξ. This means that the coordinates after a single feedback can be either anti-correlated or correlated. It is worth noting that the momenta of atoms in contrast are always correlated, which is the result of the measurement back-action on the center of mass.

Using Eq. (21) let us now address the efficiency of a single feedback step. It can be determined by calculating the average energy change due to the feedback process: $\Delta E = \langle\hat{H}\rangle_+ - \langle\hat{H}\rangle_-$, where $\langle\hat{H}\rangle_\pm$ is the average energy of atoms after/before the feedback step.

Given the average energy of the system before the feedback equals $\langle\hat{H}\rangle_- = (N\omega/2)\coth(1/\xi)$ the total energy change due to the single feedback in units of the ground state energy becomes

$$\Delta E^{(1)} = \frac{1}{2}\Big(N\sigma_1^2 + \frac{1}{N\sigma_1^2}\Big) - \frac{1}{2}\coth(1/\xi). \quad (22)$$

The positive contribution in this expression is the measurement induced noise resulting in the energy increase. The first term arises from the imprecision of the coordinate measurement and the second term is the back-action noise. The negative contribution in Eq. (22) represents the energy subtracted from the system due to the shift of the collective coordinate and, thus, represents the sought cooling effect. It is worth noting that the noise due to the measurement can be minimized choosing the optimal measurement resolution

$$\sigma_{\rm opt} = \frac{1}{\sqrt{N}}. \quad (23)$$

Consider now the effect of the second feedback step. The Wigner function of the system after the second step is obtained from the general result, Eq. (18), taking there $k=2$. The average energy change after two subsequent feedback steps with the free evolution between them during $\tau/4$ is given by

$$\Delta E^{(2)} = \frac{1}{2}\left(N\sigma_1^2 + N\sigma_2^2 + \frac{1}{N\sigma_2^2}\right) - \coth(1/\xi), \quad (24)$$

where $\sigma_{1,2}$ denotes the measurement resolution for the first and the second step, respectively. From this equation one can see that the energy removed from the system after two steps appears to be equal to the average energy of a single atom being in a thermal equilibrium state.

The noise introduced into the system contains contributions due to the measurement imprecision at both steps and the measurement back-action at the second feedback step. It is remarkable that the back-action noise due to the first step is completely compensated by the second feedback step. This is, however, true only for the specific time intervals between two feedbacks equal to $\Delta t = \tau(1+2k)/4$.

Thus, to minimize the noise it would be optimal to perform the absolutely precise measurement of the coordinate at the first step and then take the measurement resolution given by Eq. (23) at the second step. Taking this into account let us find a theoretical limit for the cooling, i.e. the temperature where the feedback does not subtract the energy from the gas. In other words at this temperature the average energy change would be zero, $\Delta E = 0$. If we let $\Delta E = 0$ in Eq. (24) then the minimal energy in units of the energy of the ground state where the feedback still works is

$$E_{\text{lim}} = \coth(1/\xi_{\text{lim}}) = 1. \tag{25}$$

That is ideally the energy can be subtracted from the gas always until all atoms are not at the ground state or, in other words, the method exhibits no theoretical cooling limits. This important result shows that the measurement back-action noise could be, in principle, overcome in an appropriately designed feedback scheme.

However, this theoretical limit can hardly be reached, since the absolutely precise measurement at the first feedback step would introduce an infinite amount of energy into the system, see the term corresponding to the back-action noise in Eq. (22). Thus, in order to keep the atoms in a laboratory until the next feedback the atoms have to be trapped inside infinitely deep potential wells, which would require in turn infinitely large intensities of the lasers forming the lattice.

Therefore, the limit energy of the gas where the feedback still works is given by

$$E_{\text{lim}} = 1 + \frac{N\sigma_1^2}{2}, \tag{26}$$

where the finite accuracy of the measurement at the first feedback is taken into account.

It is worth noting, that considered feedback cooling method contrary to the standard laser cooling techniques does not rely on resonant interactions of atoms with laser fields. Therefore, its operation is not restricted to the photon recoil limit as is the case in Doppler and Sisyphus cooling, for example. As it can be seen from Eq. (26) the minimal energy where the feedback loop still provides the cooling effect is limited only by the measurement accuracy. Except of the problems mentioned above there are no fundamental constrains forbidding the measurement resolution σ_1 to be very small. Thus, one can expect that the recoil limit can be overcome for any particular sort of atoms and laser frequencies used to cool them.

Using Eqs. (18) and (19) one can show that for the initial thermal state the phase-space volume occupied by the atoms does not change after the third feedback step. Therefore, the only energy change is the energy change due to the described above the first and the second steps. Such a behavior may be understood as a result of the appearance of feedback-generated atom-atom correlations, see discussion of Eq. (21).

Thus, aiming the practical use of the considered cooling scheme at very low temperatures one should provide a mechanism destroying the atom-atom correlations.

To conclude the scheme of feedback cooling of atoms trapped in an one-dimensional far-off-resonant optical lattice has been considered in the limit of low temperatures. The evolution of the Wigner function of the atoms due to a series of feedbacks with the free harmonic rotation between them has been analytically derived. It has been shown that in spite of the presence of quantum noise the feedback scheme subtracts energy from the gas at any temperature at least in principle. In a realistic situation the limiting temperature, where the feedback does not work is determined by the classical-type noise due to an imprecision of measurements. In any case the performance of the cooling scheme at low temperatures is limited by the feedback-induced atom-atom correlations. These correlations should be gradually destroyed in order to cool atoms to the ground state of the lattice potential.

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