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Mesoscopic wave turbulence

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We report results of simulation of wave turbulence. Both inverse and direct cascades are observed. The definition of "mesoscopic turbulence" is given. This is a regime when number of modes in a system involved in turbulence is high enough to qualitatively simulate most of processes but significantly smaller then threshold which gives us quantitative agreement with statistical description, like kinetic equation. Such regime takes place in numerical simulation, essentially finite systems etc.

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Theory of wave turbulence is developed for infinitely large system. In weakly nonlinear dispersive media the turbulence is described by a kinetic equation for squared wave amplitudes (weak turbulence). However, all real systems are finite. Computer simulation of wave turbulence also can be done only in finite system (typically in a box with periodic boundary conditions). It is important to know how strong discreteness of a system impacts the physical picture of wave turbulence.

Let a turbulence be realized in Q-dimensional cube with side L. Then wave vectors form a cubic lattice with the lattice constant $\Delta k = 2\pi/L$. Suppose that four-wave resonant conditions are dominating. Exact resonances satisfy the equations

$$\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 = 0, \tag{1}$$

$$\Delta = \omega(k) + \omega(k_1) - \omega(k_2) - \omega(k_3) = 0. \tag{2}$$

In infinite medium Eq. (1), (2) define hypersurface dimension $3Q-1$ in $4Q$-dimensional space $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. In a finite system (1), (2) are Diophantine equations which might have or have no exact solutions. The Diophantine equation for four-wave resonant processes are not studied yet. For three-wave resonant processes they are studied for Rossby waves on β-plane [1].

However, not only exact resonances are important. Individual harmonics in the wave ensemble fluctuate with inverse time $\Gamma_{\mathbf{k}}$, dependent on their wave numbers. Suppose that all $\Gamma_{\mathbf{k}_i}$ for waves, composing a resonant quartet are of the same order of magnitude $\Gamma_{\mathbf{k}_i} \sim \Gamma$. Then resonant equation (2) has to be satisfied up to accuracy $\Delta \sim \Gamma$, and the resonant surface is blurred into the layer of thickness $\delta k/k \simeq \Gamma_k/\omega_k$. This thickness should be compared with the lattice constant Δk. Three different cases are possible:

1. $\delta k \gg \Delta k$. In this case the resonant layer is thick enough to hold many approximate resonant quartets on a unit of resonant surface square. These resonances are dense, and the theory is close to the classical weak turbulent theory in infinite media. The weak turbulent theory offers recipes for calculation of Γ_k. The weak-turbulent Γ_k are the smallest among all given by theoretical models. To be sure that the case is realized, one has to use weak-turbulent formulae for Γ_k.
2. $\delta k < \Delta k$. This is the opposite case. Resonances are rarefied, and the system consists of a discrete set of weakly interacting oscillators. A typical regime in this situation is the "frozen turbulence" [2 – 4], which is actually a system of KAM tori, accomplished with a weak Arnold's diffusion.
3. The intermediate case $\delta k \simeq \Delta k$ can be called "mesoscopic turbulence". A density of approximate resonances is high enough to provide the energy transport along the spectrum, but low enough to guarantee "equal rights" for all harmonics, which is necessary condition for applicability of the weak turbulent theory.

In this article we report results of our numerical experiments on modeling of turbulence of gravity waves on the surface of deep ideal incompressible fluid. The motivation of this work was numerical justification of Hasselmann kinetic equation. The result is discovery of

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the mesoscopic turbulence. The fluid motion is potential and described by shape of surface $\eta(\mathbf{r},t)$ and velocity potential $\psi(\mathbf{r},t)$, evaluated on the surface. These variables satisfy the canonical equations [5]

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \tag{3}$$

Hamiltonian H is presented by first three term in expansion on powers of nonlinearity $\nabla\eta$

$$\begin{aligned} &H = H_0 + H_1 + H_2 + ..., \\ &H_0 = \frac{1}{2}\int \left(g\eta^2 + \psi\hat{k}\psi\right) \mathrm{d}x\mathrm{d}y, \\ &H_1 = \frac{1}{2}\int \eta\left[|\nabla\psi|^2 - (\hat{k}\psi)^2\right] \mathrm{d}x\mathrm{d}y, \\ &H_2 = \frac{1}{2}\int \eta(\hat{k}\psi)\left[\hat{k}(\eta(\hat{k}\psi)) + \eta\nabla^2\psi\right] \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4}$$

Thereafter We put gravity acceleration $g = 1$. Here $\hat{k}$ is a linear integral operator $\left(\hat{k} = \sqrt{-\nabla^2}\right)$, such that in k-space it corresponds to multiplication of Fourier harmonics ($\psi_{\mathbf{k}} = \frac{1}{2\pi}\int \psi_{\mathbf{r}} e^{i\mathbf{kr}}\mathrm{d}x\mathrm{d}y$) by $\sqrt{k_x^2 + k_y^2}$. For gravity waves this reduced Hamiltonian describes four-wave interaction. Then dynamical equations (3) acquire the form

$$\begin{aligned} \dot{\eta} = \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\ +\hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \frac{1}{2}\nabla^2[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\nabla^2\psi], \\ \dot{\psi} = -g\eta - \frac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\ -[\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\nabla^2\psi. \end{aligned} \tag{5}$$

Let us introduce the canonical variables $a_{\mathbf{k}}$ as shown below

$$a_{\mathbf{k}} = \sqrt{\frac{\omega_k}{2k}}\eta_{\mathbf{k}} + i\sqrt{\frac{k}{2\omega_k}}\psi_{\mathbf{k}}, \tag{6}$$

where $\omega_k = \sqrt{gk}$. In these so called normal variables equations (3) take the form

$$\frac{\partial a_{\mathbf{k}}}{\partial t} = -i\frac{\delta H}{\delta a^*_{\mathbf{k}}}. \tag{7}$$

The physical meaning of these variables is quite clear: $|a_{\mathbf{k}}|^2$ is an action spectral density, or $|a_{\mathbf{k}}|^2\Delta k^2$ is a number of particles with particular wave number $\mathbf{k}$.

We solved equations (5) numerically in a box $2\pi\times 2\pi$ using spectral code on rectangular grid with double periodic boundary conditions. The implicit energy-preserving scheme, similar to used in [6–8] was implemented. We studied evolution of freely propagating waves (swell) in the absence of wind in the spirit of paper [9]. Different grids (512×512, 256×1024, 256×2048) with different initial data were tried. In all cases we observed mesoscopic wave turbulence. The most spectacular results are achieved on the grid 256×2048.

As an initial conditions we used Gauss-shaped distribution on a long axis of the wavenumbers plane

$$\begin{cases} |a_{\mathbf{k}}| = A_i \exp\left(-\frac{1}{2}\frac{|\mathbf{k}-\mathbf{k}_0|^2}{D_i^2}\right), |\mathbf{k}-\mathbf{k}_0| \le 2D_i, \\ |a_{\mathbf{k}}| = 10^{-12}, |\mathbf{k}-\mathbf{k}_0| > 2D_i, \end{cases} \\ A_i = 5\cdot10^{-6}, D_i = 30, \mathbf{k}_0 = (0;150). \tag{8}$$

Initial phases of all harmonics were random. Average steepness $\mu =< |\nabla\eta| >\simeq 0.115$. To stabilize computations in high-frequency region [10] we introduced artificial damping, mimicking viscosity at small scales and artificial smoothing term to equation for surface evolution

$$\begin{aligned} &\frac{\partial\psi_{\mathbf{k}}}{\partial t} \to \frac{\partial\psi_{\mathbf{k}}}{\partial t} + \gamma_k\psi_{\mathbf{k}}, \\ &\frac{\partial\eta_{\mathbf{k}}}{\partial t} \to \frac{\partial\eta_{\mathbf{k}}}{\partial t} + \gamma_k\eta_{\mathbf{k}}, \\ &\gamma_k = \begin{cases} 0, k < k_d, \\ -\gamma(k-k_d)^2, k \ge k_d, \end{cases} \\ &k_d = 512, \gamma = 2\cdot10^4, \tau = 3.1\cdot10^{-4}. \end{aligned} \tag{9}$$

With the time step τ this calculations took about two months on AMD Athlon 64 3500+. During this time we reached 1500 periods of the wave in the initial spectral maximum.

The process of waves evolution can be separated in two steps. On the first stage (about fifty initial wave periods) we observe fast loss of energy and wave action. This effect can be explained by formation of "slave" harmonics, taking their part of motion constants. Initially smooth spectrum becomes very rough. The spectral maximum demonstrates fast downshift.

On the second stage the downshift continues, but all processes slow down. Plots of energy, wave action, mean frequency and mean steepness are presented on Fig.1–4.

One can see clear tendency to downshift of spectral maximum corresponding to inverse cascade, however this process is more slow then predicted by weak turbulence theory. Self-similar downshift in this theory gives [11, 12]

$$\omega \sim t^{-1/11}.$$

In our experiments

$$\omega \sim t^{-\alpha},$$

where α decreases with time from 1/16 to 1/20. Evolution of angle averaged spectra $N_k = \int_0^{2\pi} |a_{\mathbf{k}}|^2 k\mathrm{d}k\mathrm{d}\vartheta$ is

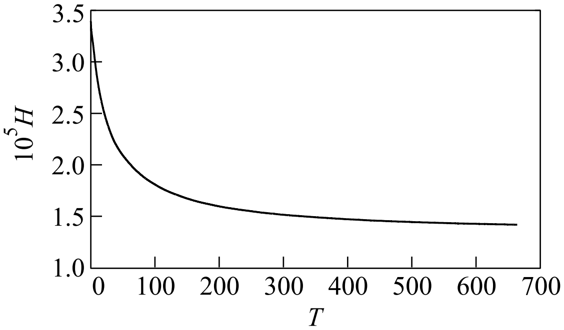

Fig.1. Total energy of the system

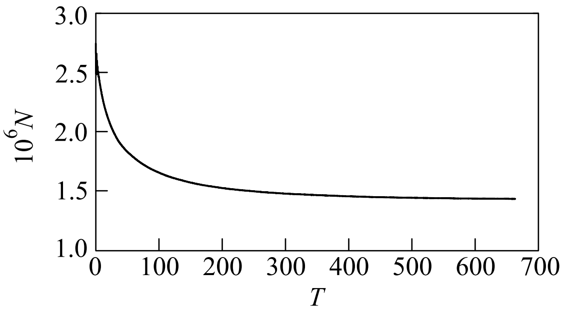

Fig.2. Total action of the system

presented on Fig.5. Their tails (Fig.6) are Zakharov–Filonenko weak-turbulent Kolmogorov spectra [13] corresponding to direct cascade

$$\langle |a_k|^2 \rangle \sim 1/k^4. \qquad (10)$$

This result is robust, it was observed in similar calculations [9, 7, 8].

Two dimensional spectra in the initial and in the last moments of calculations are presented on Fig.7. One can see formation of small intensity "jets" posed on the Phillips resonant curve [14]

$$2\omega(\mathbf{k}_0) = \omega(\mathbf{k}_0 + \mathbf{k}) + \omega(\mathbf{k}_0 - \mathbf{k}). \qquad (11)$$

Spectra are very rough and sharp. The slice of spectra along the line $(0; k_y)$ in the end of computations is presented on Fig.8. Evolution of squared wave amplitudes for a cluster of neighbour harmonics is presented on Fig.9.

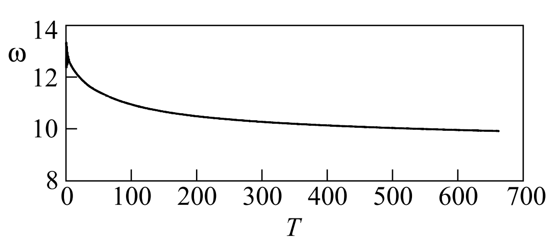

Fig.3. Frequency of the spectral maximum

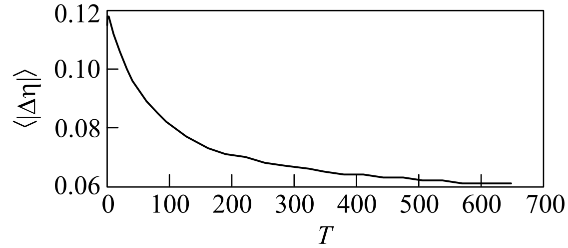

Fig.4. Mean steepness of fluid surface

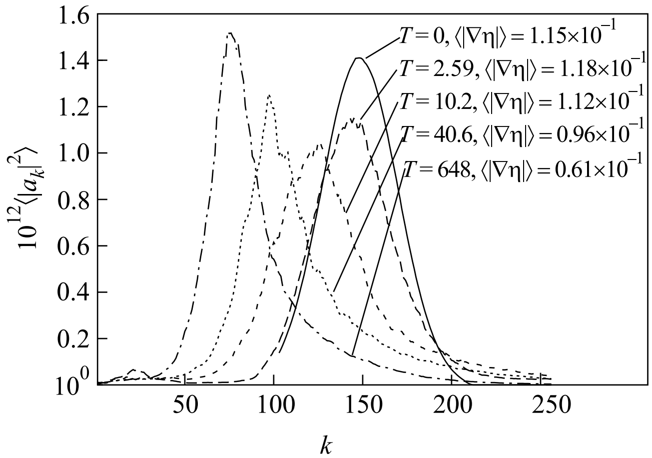

Fig.5. Averaged with angle spectra. Downshift of spectral maximum is clearly observable

Results, presented on Fig.9 show that what we modeled is mesoscopic turbulence. Indeed, characteristic time of amplitude evolution on a figure is hundred or more their periods, thus Γ/ω_k is comparable with $\Delta k/k$. On the same figure we can see the most remarkable features of such turbulence.

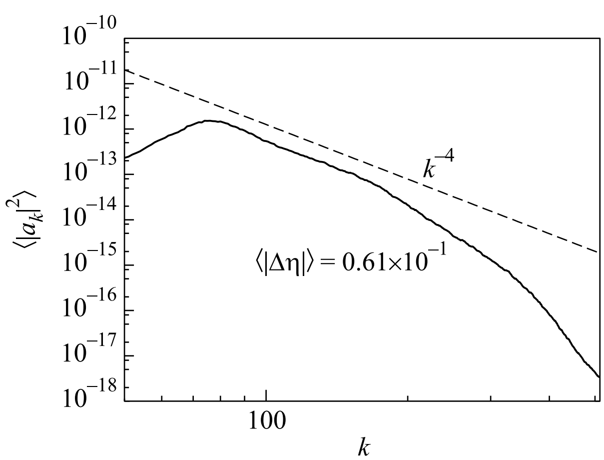

Fig.6. Tails of angle-averaged spectrum in double logarithmic scale. $T = 648 = 1263T_0$. Power-like tail and front slope are close to predicted by weak turbulent theory

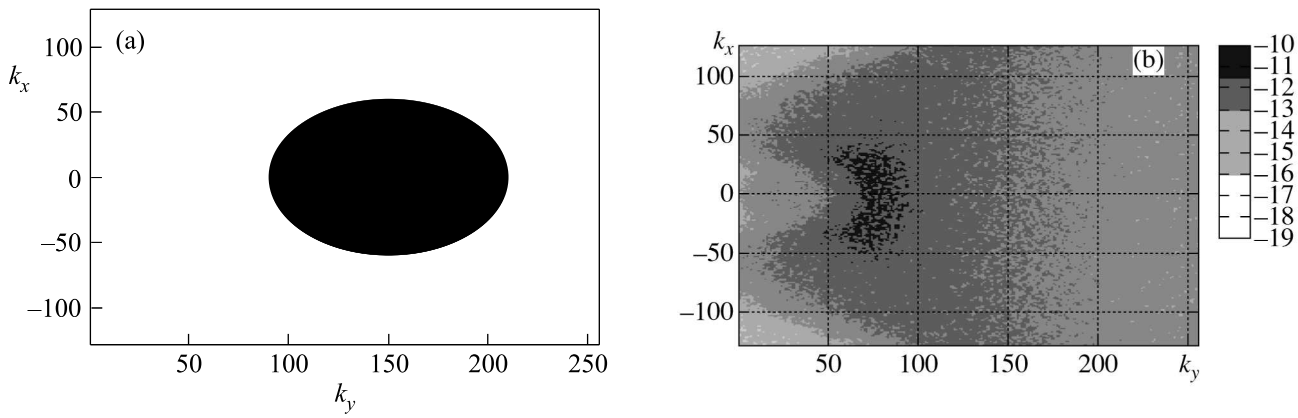

Fig.7. (a) – Level lines of logarithm of initial spectra distribution. $T = 0$. (b) – Level lines of logarithm of spectra distribution at $T = 648 = 1263T_0$

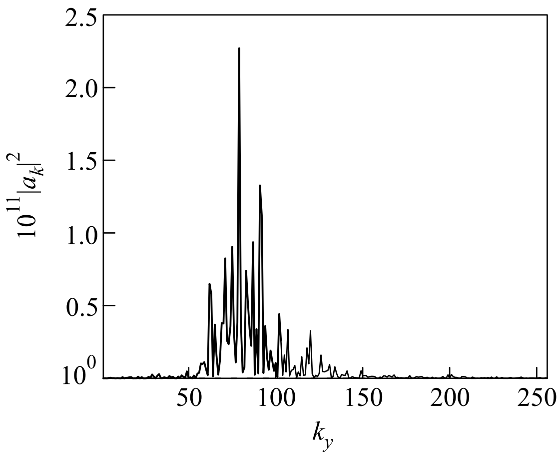

Fig.8. Slice of spectrum on axis $(0; k_y)$ at $T = 648 = 1263T_0$

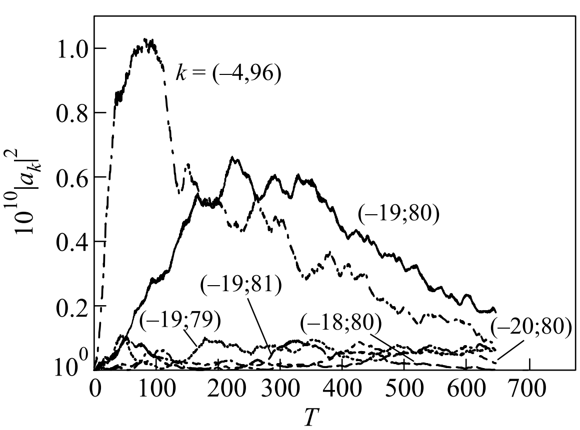

Fig.9. Evolution of some cluster of harmonics and a distant large harmonic

The weak turbulence in the first approximation obeys the Gaussian statistics. Neighbour harmonics are uncorrelated and statistically independent ($\langle a_k a^*_{k+1} \rangle = 0$). However their averaged characteristics are close to each other. This is a "democratic society". On the contrary mesoscopic turbulence is the "oligarchic society". The Phillips curve (11) has a genus 2. After Faltings' proof [15] of Mordell's hypothesis [16] we know that the number of solutions of the Diophantine equation

$$\Delta = 2(n^2 + m^2)^{1/4} - [(n + x)^2 + (m + y)^2]^{1/4} - - [(n - x)^2 + (m - y)^2]^{1/4} = 0 \qquad (12)$$

is at most finite and most probably, except few trivial solutions, equals to zero. The same statement is very plausible for more general resonances. Approximate integer solutions in the case

$$|\Delta| < \epsilon$$

do exist, but their number fast tends to zero at $\epsilon \to 0$. Classification of these solutions is a hard problem of the number theory. These solutions compose the "elite society" of the harmonics, which play the most active role in the mesoscopic turbulence. Almost all inverse cascade of wave action is realized within members of this "privileged club". The distribution of harmonics exceeding a reference level $|a_k|^2 = 10^{-11}$ at the moment $t = 1200T_0$ is presented on Fig.10. A number of such harmonics is not more than 600, while a total number of harmonics involved into the turbulence is of the order of 10^4.

Note that a situation with direct cascade is different. As far as the coupling coefficient for gravity waves growth as fast as k^3 with wave number, for short waves Γ_k/ω_k easily exceeds $\Delta k/k$, and conditions of applicability of the weak turbulent theory for short waves are satisfied.

Note also that the mesoscopic turbulence is not a numerical artefact. Simple estimations show that for gravity waves it is realized in some conditions in basins of a

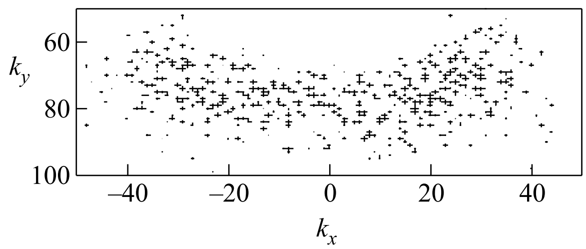

Fig.10. Harmonics with square modulus exceeding level 10^{-11} at $T = 648 = 1263T_0$

moderate size, like small lakes as well as in experimental wave tanks. It is also common for long internal waves in ocean and for inertial gravity waves in atmosphere, for plasma waves in tokamaks etc.

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