Pis'ma v ZhETF, vol. 82, iss. 8, pp. 603 – 607 © 2005 October 25

Universal statistics of the local Green function in wave chaotic systems with absorption

*D. V. Savin, H.-J. Sommers, Y. V. Fyodorov**+

Fachbereich Physik, Universität Duisburg-Essen, 45117 Essen, Germany

**School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom*

\+ *Petersburg Nuclear Physics Institute RAS, 188300 Gatchina, Russia*

Submitted 19 September 2005

We establish a general relation between the statistics of the local Green function for systems with chaotic wave scattering and uniform energy loss (absorption) and the two-point correlator of its resolvents for the same system without absorption. Within the random matrix approach this kind of a fluctuation dissipation relation allows us to derive the explicit analytic expression for the joint distribution function of the real and imaginary part of the local Green function for all symmetry classes as well as at an arbitrary degree of time-reversal symmetry breaking in the system. The outstanding problem of orthogonal symmetry is further reduced to simple quadratures. The results can be applied, in particular, to the experimentally accessible impedance and reflection in a microwave cavity attached to a single-mode antenna.

PACS: 05.45.Mt, 73.23.–b, 42.25.Bs

Statistical fluctuations of physical observables in quantum systems with underlying chaotic classical dynamics are the subject of a very active field of research in theoretical and experimental physics. A considerable progress was underpinned by revealing the apparent universality of the fluctuations in systems of very diverse microscopic nature, ranging from atomic nuclei and Rydberg atoms in strong external fields, to complex molecules, quantum dots, and mesoscopic samples [1, 2]. This universality allows one to exploit the random matrix theory (RMT) as a powerful tool for a theoretical analysis of generic statistical properties of such systems [3]. In many cases the quantity, which is readily obtained experimentally, is the absorption spectrum for transitions from a given initial state to highly excited chaotic states at the energy E. Most frequently the absorption spectra look practically continuous due to both inevitable level broadening and finite experimental resolution. Then the relevant statistics are the distribution and the correlation functions of the absorption probability [4]. In the simplest situation of uniform level broadening Γ, the problem thus amounts to studying statistical properties of the resolvent (Green function) operator $\widehat{G}(E) \equiv (E + i\Gamma/2 - \widehat{H})^{-1}$ associated with the random matrix $\widehat{H}$, which replaces the actual chaotic Hamiltonian. In particular, the imaginary part of diagonal entries of $\widehat{G}(E)$ is well known in solid state physics as the *local density of states* (LDoS) and in this capacity its statistics enjoyed many studies [4 – 8].

From the experimental point of view the same universality, which makes the use of the RMT legitimate, provides one with an attractive possibility to employ simple model systems for analyzing generic statistics of the fluctuating quantities. Various billiards are nowadays proved to be an ideal playground for investigating a variety of quantum chaos phenomena; see Ref. [2] for a general discussion as well as [9 – 13] for the current problematic. They are usually realized as electromagnetic resonators in a form of two-dimensional cavities shaped to ensure the chaoticity of internal scattering and coupled to waveguides (antennas), which are used to inject microwaves into the system as well as to collect the output. In particular, it turns out that for a single-mode antenna the local Green function $G(E)$ (i.e. a diagonal element of $\widehat{G}$ taken at the port position) has the direct physical meaning of the electric impedance of the cavity which relates linearly voltages and currents at the antenna port; see [10] for a discussion. In this way not only the imaginary part but also the real part of G turn out to be experimentally accessible quantities whose statistics is, therefore, of considerable interest. Inevitable energy losses (absorption) must be taken into account properly when describing the experiments theoretically [14, 15].

The majority of the experiments is performed in systems which are time-reversal invariant (the so-called orthogonal symmetry class of the RMT characterized by Dyson's symmetry index $\beta = 1$). However, nonperturbative analytical results are available presently

only for systems with no time-reversal symmetry (TRS) ($\beta = 2$ or unitary symmetry class) derived by various methods in [5, 8], and for systems with spin-orbit scattering ($\beta = 4$ or symplectic symmetry class) [8]. An attempt [6] to provide an expression for the LDoS distribution for the $\beta = 1$ case can not be considered as particularly successful, as the final expression was given in a form of an intractable fivefold integral. Very recently a heuristic interpolation formula at $\beta = 1$ incorporating exactly the limiting cases of strong and weak absorption was suggested [8, 9] to describe reasonably well the available data at moderate absorption. Therefore, an exact analytical treatment of the $\beta = 1$ case remains a considerable theoretical challenge.

In this Letter we present a novel approach to the problem which allows us to derive the joint distribution function of the real and imaginary parts of the local Green function exactly at arbitrary absorption for the whole regime of the gradual TRS breaking between the $\beta = 1$ and $\beta = 2$ symmetry classes.

The Hamiltonian $\widehat{H}$ of the chaotic system gives rise to N levels (eigenfrequencies) characterized locally in the relevant range of the energy E by the mean level spacing Δ. We consider, as usual, dimensionless quantities expressed in units of Δ, $K(E) \equiv (N\Delta/\pi)G(E)$, and define the distribution of interest as follows:

$$\mathcal{P}(u,v) = \langle \delta(u - \mathrm{Re}\, K)\, \delta(v + \mathrm{Im}\, K) \rangle \ . \qquad (1)$$

Angular brackets stand for the ensemble averaging. In such units the mean LDoS $\langle v \rangle = 1$. The function $iK = Z$ has also the meaning of the normalized cavity impedance Z [10].

We start with establishing the general relation between the joint distribution function (1) at *finite* absorption (assumed to be uniform, $\Gamma > 0$) and the energy autocorrelation function

$$C_\Omega(z_-, z_+) = \left\langle \frac{1}{z_- - i0 - K_0(E - \Omega/2 - i0)} \times \right.$$
$$\left. \times \frac{1}{z_+ + i0 - K_0(E + \Omega/2 + i0)} \right\rangle \qquad (2)$$

of the *resolvents* of the local Green function K_0 at *zero* absorption ($\Gamma = 0$). Distribution (1) can be obtained from (2) by analytic continuation in Ω from a real to purely imaginary value $\Omega = i\Gamma$ as follows. $K_0(E)$ is an analytic function of the energy in the upper or lower half-plane and can be thus analytically continued to the complex values: $K_0(E \pm i\Gamma/2) \equiv u \mp iv$, $v > 0$. This allows us to continue then analytically the correlation function (2) from a pair of its real arguments to the complex conjugate one: $z_+ = (z_-)^* \equiv z' + iz''$, $z'' > 0$. As a result, function (2) acquires at $\Omega = i\Gamma$ the following form:

$$C(z', z'') \equiv C_{\Omega = i\Gamma}(z_-, z_+) = \left\langle \frac{1}{(z'-u)^2 + (z''+v)^2} \right\rangle =$$
$$= \int_{-\infty}^{\infty} du \int_0^{\infty} dv\, \frac{\mathcal{P}(u,v)}{(z'-u)^2 + (z''+v)^2}. \qquad (3)$$

The second line here is due to the definition (1). To solve this equation for $\mathcal{P}(u,v)$, we perform first the Fourier transform (FT) $\widehat{C}(k, z'') \equiv \int_{-\infty}^{\infty} dz' e^{ikz'} C(z', z'')$ with respect to z' that leads to

$$\widehat{C}(k, z'') = \int_0^{\infty} dv\, \widehat{\mathcal{P}}(k, v)\, \frac{\pi\, e^{-|k|(z''+v)}}{z'' + v}, \qquad (4)$$

where $\widehat{\mathcal{P}}(k,v)$ is the corresponding FT of $\mathcal{P}(u,v)$. Being derived at $z'' > 0$, Eq. (4) can be analytically continued to the whole complex z'' plane with a cut along negative $\mathrm{Re}\, z''$. Calculating then the jump of $\widehat{C}(k, z'')$ on the discontinuity line $z'' = -v$ ($v > 0$), we finally get the following expression

$$\widehat{\mathcal{P}}(k,v) = \frac{\Theta(v)}{2\pi^2 i}[\widehat{C}(k, -v - i0) - \widehat{C}(k, -v + i0)], \qquad (5)$$

with the Heaviside step function $\Theta(v)$. The inverse FT of (5) yields the desired distribution.

This relationship is one of our central results. It resembles (and reduces to) the well-known relation between the spectral density of states and the imaginary part of the corresponding resolvent operator when the case of one real variable is considered. In contrast, the case of the distribution function of two real variables requires to deal with the two-point correlation function. Physically, the latter is a generalized susceptibility which describes a response of the system that allows one to treat (5) in the sense of a fluctuation dissipation relation: The l.h.s. there stands for the *distribution* (of K) in the presence of dissipation / absorption whereas the *correlator* (of resolvents of K) in the r.h.s. accounts for fluctuations in the system, i.e. for arbitrary order correlations in the absence of absorption.

We proceed now with applications. The main advantage of the derived relation is that the correlator is a much easier object to calculate analytically than the distribution and such a calculation for ideal systems at zero absorption has actually already been performed in many interesting cases. Let us consider the chaotic cavity mentioned already. In this case an exact result for the correlation function (2) has been previously obtained

by us in Refs. [16, 17]. Its analytic continuation to complex $\Omega = i\Gamma$ can be represented generally as follows:

$$C(z', z'') = \frac{1}{z'^2 + (z''+1)^2} + \frac{1}{4}\left(\frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{\partial z''^2}\right)\mathcal{F}(\tilde{x}), \tag{6}$$

where it is important that the function $\mathcal{F}(\tilde{x})$ depends on z' and z'' only via the scaling variable $\tilde{x} \equiv (z'^2 + z''^2 + 1)/2z'' > 1$. Its explicit form depends on the symmetry present (e.g. preserved or broken TRS), the following common structure being however generic:

$$\mathcal{F}(\tilde{x}) = \int_{-1}^{1} d\lambda_0 \int_{1}^{\infty} d\lambda_1 \int_{1}^{\infty} d\lambda_2\, f(\{\lambda\})\, e^{-\gamma(\lambda_1\lambda_2 - \lambda_0)/2} \times \times \left[\frac{(\tilde{x}+\lambda_0)^2}{(\tilde{x}+\lambda_1\lambda_2)^2 - (\lambda_1^2-1)(\lambda_2^2-1)}\right]^{1/2}. \tag{7}$$

Here, the dimensionless parameter $\gamma \equiv 2\pi\Gamma/\Delta$ (i.e. absorption width Γ in units of the mean level spacing Δ) accounts for the absorption strength. The *real* function $f(\{\lambda\})$ is the only symmetry dependent term. In the crossover regime of gradually broken TRS it can be represented explicitly as follows [17]:

$$f(\{\lambda\}) = \{(1-\lambda_0^2)(1+e^{-2Y}) - (\lambda_1^2 - \lambda_2^2)(1-e^{-2Y}) + 4y^2\mathcal{R}[(1-\lambda_0^2)e^{-2Y} + \lambda_2^2(1-e^{-2Y})]\}\frac{e^{-2y^2(\lambda_2^2-1)}}{\mathcal{R}^2}, \tag{8}$$

with $\mathcal{R} = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda_0\lambda_1\lambda_2 - 1$ and $Y \equiv y^2(1-\lambda_0^2)$, where y denotes a crossover driving parameter. Physically, $y^2 \sim \delta E_y/\Delta$ is determined by the energy shift δE_y of energy levels due to a TRS breaking perturbation (e.g., weak external magnetic field in the case of quantum dots). Such an effect is conventionally modelled within the framework of RMT by means of the "Pandey-Mehta" Hamiltonian [18], $\widehat{H} = \widehat{H}_S + i(y/\sqrt{N})\widehat{H}_A$, with $\widehat{H}_S$ ($\widehat{H}_A$) being a random real symmetric (antisymmetric) matrix with independent Gaussian distributed entries. The limit $y \to 0$ or ∞ corresponds to fully preserved or broken TRS, respectively.

Now we apply relation (5) to Eq. (6) and then perform the inverse FT to get $\mathcal{P}(u, v)$. Relegating all technical details to a more extended publication, we emphasize the most important points. The nontrivial contribution to the distribution comes from the second ("connected") part of the correlation function (6) whereas the first ("disconnected") one is easily found to yield the singular contribution $\delta(u)\delta(v-1)$. A careful analysis shows that due to specific $\tilde{x}$-dependence given by Eq. (7) the above described procedure for the analytic continuation of the connected part of $\widehat{C}(k, z'')$ is equivalent to continuing $\mathcal{F}(\tilde{x})$ analytically and taking the jump at $\tilde{x} = -x \pm i0$, with

$$x \equiv \frac{u^2 + v^2 + 1}{2v} > 1.$$

The nonzero imaginary part $F(x) = \mathrm{Im}\,\mathcal{F}(-x)$ is thus determined at given x by the integration region $\mathcal{B}_x = \{(\lambda_1, \lambda_2) \mid 1 \le \lambda_{1,2} < \infty,\ (\lambda_1\lambda_2 - x)^2 < (\lambda_1^2 - 1)(\lambda_2^2 - 1)\}$, where the square root in (7) attains pure imaginary values. Taking into account the identity $\partial^2 F(x)/\partial u^2 + \partial^2 F(x)/\partial v^2 = v^{-2}\frac{d}{dx}(x^2-1)\frac{d}{dx}F(x)$ valid for $x^2 \neq 1$, we arrive finally at

$$\mathcal{P}(u, v) = \frac{1}{4\pi^2 v^2}\frac{d}{dx}(x^2-1)\frac{dF(x)}{dx} \equiv \frac{1}{2\pi v^2}P_0(x). \tag{9}$$

This distribution is easily checked to be invariant under the change $iK \to 1/iK$, meaning physically that the impedance and its inverse have one and the same distribution function.

Such a form of the distribution is completely generic, as all the symmetry specific dependencies were not essential for the above discussion. It can be shown [8] by exploiting the well-known relation $S = (1 - i\kappa K)/(1 + i\kappa K) \equiv \sqrt{r}e^{i\theta}$ between the scattering matrix S and the local Green function (known as K function in this context), that the representation given by the second equality in Eq. (9) is a consequence of the two following properties at the so-called *perfect coupling*, $\kappa = 1$ [19]: (i) the uniform distribution of the scattering phase $\theta \in (0, 2\pi)$; and (ii) the statistical independence of θ and the S matrix modulus. This establishes also a physical meaning of x by relating it to the *reflection coefficient* r, thus $P_0(x)$ being the normalized distribution of $x = (1+r)/(1-r)$. Remarkably, Eq. (9) relates the distribution of the local Green function in the closed system to that of reflection in the perfectly open one.

Both these properties can be verified using the methods of Ref. [14] but only in the cases of preserved or completely broken TRS. Our approach proves it generally for the crossover regime at an arbitrary degree of TRS breaking. Taking into account our findings, we can bring the final result to the following form:

$$F(x) = \int_{-1}^{1} d\lambda_0 \iint_{\mathcal{B}_x} d\lambda_1 d\lambda_2\, f(\{\lambda\})\, e^{-\gamma(\lambda_1\lambda_2 - \lambda_0)/2} \times \times \frac{(x - \lambda_0)}{[(\lambda_1^2-1)(\lambda_2^2-1) - (\lambda_1\lambda_2 - x)^2]^{1/2}}. \tag{10}$$

At arbitrary values of the crossover parameter y the obtained result can be treated only numerically. Further analytical study is possible in the limiting cases of pure symmetries considered below.

The simplest case of unitary symmetry ($\beta = 2$) [20] is correctly reproduced from (10) as $y \to \infty$. We find [21]:

$$P_0(x) = \frac{\mathcal{N}_\beta}{2}\left[A\ (\alpha(x+1)/2)^{\beta/2} + B\right] e^{-\alpha(x+1)/2}, \tag{11}$$

where it is convenient for the subsequent use to introduce the absorption parameter $\alpha \equiv \gamma\beta/2$ scaled with the symmetry index β, α-dependent constants being $A \equiv e^\alpha - 1$ and $B \equiv 1 + \alpha - e^\alpha$, and the normalization constant $\mathcal{N}_2 = 1$.

As to the case of orthogonal symmetry ($\beta = 1$), no general result was available in the literature. Expression (11) (with $\mathcal{N}_\beta = \alpha/(A\Gamma(\beta/2+1,\alpha) + Be^{-\alpha})$ and $\Gamma(\nu,\alpha) = \int_\alpha^\infty dt\, t^{\nu-1}e^{-t}$) was suggested in Ref. [8] (see also [9]) to be an appropriate interpolation formula at $\beta = 1$. It incorporates correctly both known limiting cases of weak or strong absorption and a reasonable agreement with available numerical and experimental data was found in a broad range of the absorption strength. We proceed with providing an exact analytical treatment of this case which amounts to investigating (8) and (10) at $y = 0$. Fortunately, further simplifications are possible if one considers the integrated probability distribution

$$W(x) \equiv -\frac{x^2-1}{2\pi}\frac{dF(x)}{dx} = \int_x^\infty dx\, P_0(x), \tag{12}$$

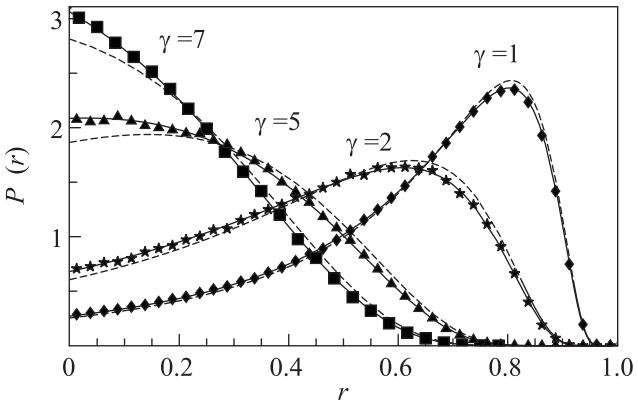
which is a positive monotonically decaying function by definition. To this end, we note that it is useful to switch to the parametrization of Ref. [22] to carry out the threefold integration. The latter turns out to yield then a sum of decoupled terms and, after some algebra, we have been able to cast the result in the following final form:

$$W(x) = \frac{x+1}{4\pi}\Big[f_1(w)g_2(w) + f_2(w)g_1(w) + \\ + h_1(w)j_2(w) + h_2(w)j_1(w)\Big]_{w=(x-1)/2}, \tag{13}$$

with auxiliary functions defined as follows:

$$f_1(w) = \int_w^\infty dt \frac{\sqrt{t|t-w|}\, e^{-\gamma t/2}}{(1+t)^{3/2}}[1 - e^{-\gamma} + t^{-1}],$$
$$g_1(w) = \int_w^\infty dt \frac{1}{\sqrt{t|t-w|}}\frac{e^{-\gamma t/2}}{(1+t)^{3/2}},$$
$$h_1(w) = \int_w^\infty dt \frac{\sqrt{|t-w|}\, e^{-\gamma t/2}}{\sqrt{t(1+t)}}[\gamma + (1-e^{-\gamma})(\gamma t - 2)],$$
$$j_1(w) = \int_w^\infty dt \frac{1}{\sqrt{t|t-w|}}\frac{e^{-\gamma t/2}}{\sqrt{1+t}}, \tag{14}$$

their counterpart with the index 2 being given by the same expression save for the integration region $t \in [0, w]$ instead of $[w, \infty)$. For an illustration of our findings the reflection distribution $P(r) = \frac{2}{(1-r)^2}P_0(\frac{1+r}{1-r})$ is shown on Figure.



The reflection coefficient distribution in chaotic systems invariant under time-reversal at perfect coupling is plotted at moderate values of the absorption strength $\gamma = 2\pi\Gamma/\Delta$, where systematic deviations between the exact result drawn from Eq. (13) (solid lines) and the interpolation expression (11) at $\beta = 1$ (dashed lines) are most noticeable. Symbols stand for numerics done for 10^3 realizations of 500×500 random GOE matrices

Finally, we mention that the general case of arbitrary transmission $T < 1$ can be mapped [23, 24] onto that of perfect one considered so far. The scattering phase θ is then no longer uniformly distributed and gets statistically correlated with x. However, their joint distribution $P(x,\theta)$ can be found [8] to be again determined by $P_0(x)$ as follows:

$$P(x,\theta) = \frac{1}{2\pi}P_0(xg - \sqrt{(x^2-1)(g^2-1)}\cos\theta), \tag{15}$$

with $g \equiv 2/T - 1$. This equation provides us further with distributions of the phase and reflection coefficient which were recently studied experimentally [9].

In summary, we provided a new approach to statistical properties of the local Green function in quantum chaotic or disordered absorptive systems of any symmetry class. It would be highly interesting to extend current experimental studies [11] of the crossover regime to check our findings. Although the validity of explicit formulas given above is restricted to the completely ergodic ("zero-dimensional") case, it is actually possible to adopt the above method [25] to quasi-one (or higher) dimensional situations when Anderson localization effects play already an important role [7].

We thank V.V. Sokolov for useful comments. The financial support by the SFB/TR 12 der DFG (D.V.S.

and H.-J.S.) and EPSRC grant EP/C515056/1 "Random Matrices and Polynomials: a tool to understand complexity" (Y.V.F.) is acknowledged.

1. F. Haake, *Quantum Signatures of Chaos*, Springer-Verlag, Berlin, 1991.
2. H.-J. Stöckmann, *Quantum Chaos: An Introduction*, Cambridge University Press, Cambridge, UK, 1999.
3. T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
4. Y. V. Fyodorov and Y. Alhassid, Phys. Rev. A **58**, R3375 (1998).
5. K. B. Efetov and V. N. Prigodin, Phys. Rev. Lett. **70**, 1315 (1993); C. W. J. Beenakker, Phys. Rev. B **50**, 15170 (1994); A. D. Mirlin and Y. V. Fyodorov, Europhys. Lett. **25**, 669 (1994); A. V. Andreev and B. D. Simons, Phys. Rev. Lett. **75**, 2304 (1995).
6. N. Taniguchi and V. N. Prigodin, Phys. Rev. B **54**, R14305 (1996).
7. A. D. Mirlin, Phys. Rep. **326**, 259 (2000).
8. Y. V. Fyodorov and D. V. Savin, JETP Lett. **80**, 725 (2004).
9. U. Kuhl et al., Phys. Rev. Lett. **94**, 144101 (2005).
10. S. Hemmady et al., Phys. Rev. Lett. **94**, 014102 (2005); X. Zheng, T. M. Antonsen, and E. Ott, e-prints cond-mat/0408327 and cond-mat/0408317.
11. Y.-H. Kim et al., Phys. Rev. Lett. **94**, 036804 (2005); S.-H. Chung et al., Phys. Rev. Lett. **85**, 2482 (2000).
12. O. I. Lobkis, I. S. Rozhkov, and R. L. Weaver, Phys. Rev. Lett. **91**, 194101 (2003).
13. J. Barthélemy, O. Legrand, and F. Mortessagne, Phys. Rev. E **71**, 016205 (2005).
14. P. W. Brouwer and C. W. J. Beenakker, Phys. Rev. B **55**, 4695 (1997).
15. D. V. Savin and H.-J. Sommers, Phys. Rev. E **68**, 036211 (2003); Y. V. Fyodorov, JETP Lett. **78**, 250 (2003).
16. Y. V. Fyodorov and H.-J. Sommers, J. Math. Phys. **38**, 1918 (1997).
17. Y. V. Fyodorov, D. V. Savin, and H.-J. Sommers, Phys. Rev. E **55**, R4857 (1997).
18. A. Pandey and M. L. Mehta, Commun. Math. Phys. **16**, 2655 (1983); A. Altland et al., Ann. Phys. (N.Y.) **219**, 148 (1992).
19. The coupling constant $\kappa > 0$ enters the S matrix statistics only through the so-called *transmission coefficient* $T \equiv 1 - |\langle S \rangle|^2$ $(= 4\kappa/(1+\kappa)^2$ at $E = 0)$, refering $T = 1$ (0) to the perfectly open (closed) system; see e.g. Refs. [16, 22] for details.
20. C. W. J. Beenakker and P. W. Brouwer, Physica E **9**, 463 (2001).
21. $4y^2 \int_1^\infty d\lambda_2 \lambda_2^2 e^{-2y^2(\lambda_2^2-1)} \to 1$ as $y \to \infty$, so that one can put λ_2=1 everywhere else in (10). As a result, the λ_1 integration in $F(x)$ shrinks to a point $\propto \delta(\lambda_1 - x)$, then the integration over λ_0 is easily performed to yield Eq. (11).
22. J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. **129**, 367 (1985).
23. P. A. Mello, P. Pereyra, and T. Seligman, Ann. Phys. **161**, 254 (1985); P. W. Brouwer, Phys. Rev. B **51**, 16878 (1995). The following map $S_{T=1} = (S - \sqrt{1-T})/(1 - \sqrt{1-T}S)$ relates the S matrices at perfect and arbitrary transmission thus the corresponding statistics as well; see also Ref. [9].
24. D. V. Savin, Y. V. Fyodorov, and H.-J. Sommers, Phys. Rev. E **63**, 035202 (2001).
25. Y. V. Fyodorov, D. V. Savin, and H.-J. Sommers, e-print cond-mat/0507016.