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The hydraulic jump as a white hole

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In the geometry of the circular hydraulic jump, the velocity of the liquid in the interior region exceeds the speed of capillary-gravity waves (ripplons), whose spectrum is 'relativistic' in the shallow water limit. The velocity flow is radial and outward, and thus the relativistic ripplons cannot propagate into the interior region. In terms of the effective 2+1 dimensional Painlevé-Gullstrand metric appropriate for the propagating ripplons, the interior region imitates the white hole. The hydraulic jump represents the physical singularity at the white-hole horizon. The instability of the vacuum in the ergoregion inside the circular hydraulic jump and its observation in recent experiments on superfluid ^{4}He by Rolley, Guthmann, Pettersen and Chevallier [3] are discussed.

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1. Introduction. Starting with the pioneering acoustic black hole [1], there appeared many suggestions to simulate the black and white holes in various laboratory systems (see review paper [2] and references therein). Here we discuss the most perspective analog, which has been actually realized in recent experiments with superfluid ^{4}He [3]: the circular hydraulic jump in superfluid ^{4}He simulates the 2+1 dimensional white hole for the surface waves with "relativistic" spectrum in the shallow liquid.

In Sec. 2 we discuss the effective space-time emerging for the surface waves – ripplons – in the shallow water limit. In Sec. 3 we introduce interaction of ripplons with the walls. The walls provide the absolute reference frame. In the region where the flow of the liquid with respect to this frame exceeds Landau critical velocity for ripplon radiation, the surface of the liquid becomes unstable. For the relativistic ripplons the boundary of this region serves as analog of a black-hole or white-hole horizon. The instability of the liquid towards generation of ripplons inside the horizon is the main mechanism of the decay of this 2+1 dimensional analog of the black or white hole. Similar instability of the vacuum inside the astronomical black hole is possible. In Sec. 4 we show that the hydraulic jump is the realization of the white hole horizon for the relativistic ripplons in normal liquids, In Sec. 5 the discussion is extended for the hydraulic jump in superfluids in relation to the recent experiment [3]. Some open questions require further investigations.

2. Effective metric for ripplons. The general dispersion relation $\omega(\mathbf{k})$ for ripplons – the waves on the surface of a liquid – is

$$M(k)(\omega - \mathbf{k}\cdot\mathbf{v})^2 = \rho g + k^2\sigma. \quad (1)$$

Here σ is the surface tension; ρ is mass density of the liquids; ρg is the gravity force; and $\mathbf{v}$ is the velocity of the liquid along the surface. The quantity $M(k)$ is the k-dependent mass of the liquid which is forced into motion by the oscillating surface:

$$M(k) = \frac{\rho}{k\,\tanh\,kh}, \quad (2)$$

where h is the thicknesses of the layer of the liquid.

The spectrum (1) becomes "relativistic" in the shallow water limit $kh \ll 1$, $k \ll k_0$:

$$(\omega - \mathbf{k}\cdot\mathbf{v})^2 = c^2k^2 + c^2k^4\left(\frac{1}{k_0^2} - \frac{1}{3}h^2\right), \qquad c^2 = gh \ , \ k_0^2 = \rho g/\sigma. \quad (3)$$

If the k^4 corrections are ignored, the spectrum of ripplons in the $k \to 0$ limit is described by the effective metric [4]

$$g^{\mu\nu}k_\mu k_\nu = 0, \quad k_\mu = (-\omega, k_x, k_y), \quad (4)$$

with the following elements

$$g^{00} = -1\ , \ g^{0i} = -v^i, \quad g^{ij} = c^2\delta^{ij} - v^i v^j. \quad (5)$$

The interval describing the effective 2+1 space-time in which ripplons propagate along geodesics and the corresponding covariant components of the effective metric are

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad g_{00} = -1 + \frac{v^2}{c^2}, \qquad g_{0i} = -\frac{v^i}{c^2}, \quad g_{ij} = \frac{1}{c^2}\delta_{ij}. \quad (6)$$

As distinct from the original acoustic metric introduced by Unruh [1], here c is the speed of gravity waves. It is typically much smaller than the speed of sound, which allows us to avoid different hydrodynamic instabilities inherent to the acoustic analogs of the horizon.

The spectrum (1) is valid for the prefect fluid, where dissipation due to friction and viscosity is neglected, and it must be modified when the dissipation is added. For the ripplons propagating at the interface between two superfluids the dissipation leads to a simple extra term on the right-hand side of Eq. (1) [5, 6]:

$$M(k)(\omega - \mathbf{k}\cdot\mathbf{v})^2 = \rho g + k^2\sigma - i\Gamma\omega. \qquad (7)$$

For the ripplons at the interface between ^{3}He-A and ^{3}He-B the friction parameter $\Gamma > 0$ depends on temperature and is proportional to T^3 at low T. The important property of the added dissipative term is that it introduces the reference frame of the horizontal wall. The ω-dependence of the dissipative term in Eq. (7), which has no Doppler shift, implies that this spectrum is written in the frame of the wall. We may expect that under some conditions this description is applicable to the normal viscous liquid, where the phenomenological parameter Γ is determined by Reynolds number of the flowing liquid and probably depends on ω and k.

3. Instability in the ergoregion. If the non-zero Γ is taken into account, from the spectrum $\omega(k)$ in Eq. (7) it follows that the instability to the formation of the surface waves occurs when the velocity v of the flow with respect to the wall exceeds the critical velocity v_L. At $v = v_L$ the imaginary part Im $\omega(k_c)$ of the energy spectrum of the critical ripplon with momentum k_c crosses zero and becomes positive, i.e. the attenuation of ripplons at $v < v_L$ due to dissipation transforms to amplification at $v > v_L$ [5, 6]. The critical velocity v_L and the momentum of the critical ripplon k_c do not depend on the friction parameter Γ. They are different in the "relativistic" and "non-relativistic" regimes:

$$v_L = c, \quad k_c = 0, \quad \text{if} \quad hk_0 < \sqrt{3}, \qquad (8)$$

$$v_L = c\sqrt{2/hk_0}, \quad k_c = k_0, \quad \text{if} \quad hk_0 \gg 1. \qquad (9)$$

In both regimes the frequency of the critical ripplon is $\omega(k_c) = 0$, i.e. the critical ripplon must be stationary in the wall frame.

The fact that the threshold velocity v_L does not depend on Γ demonstrates that the main role of the dissipative term is to provide the reference frame of the wall with which the liquid interacts. The flow of a superfluid liquid with respect to this reference frame does not experience any dissipation if its velocity is below v_L. The dissipation starts above the instability threshold when the surface of the liquid is perturbed, i.e. ripplons are radiated due interaction of the liquid with the wall. This indicates that the critical velocity of the flow with respect to the wall coincides with the Landau criterion for ripplon nucleation:

$$v_L = \min_k \frac{E(k)}{k}, \quad E(k) = \sqrt{(\rho g + \sigma k^2)/M(k)}. \qquad (10)$$

In the case of the interface between ^{3}He-A and ^{3}He-B, the critical velocity of instability towards the growth of critical ripplon has been measured in the nonrelativistic deep-water regime $hk_0 \gg 1$ [7], and has been found in a good agreement with the theoretical estmate of the Landau velocity (modified for the case of two liquids [5, 6]) wihout any fitting parameter.

The region, where the flow velocity v exceeds v_L, represents the ergoregion, since in the wall frame the energy of the critical ripplon is negative in this region, $E(k) + \mathbf{k}\cdot\mathbf{v} < 0$. For the relativistic ripplons, the ergoregion – the region where v exceeds c – is expressed in terms of the effective metric in Eq. (6): in the ergoregion the metric element g_{00} changes sign and becomes positive. If the flow is perpendicular to the ergosurface (the boundary of the ergoregion), then the ergosurface serves as the event horizon for ripplons. It is the black hole horizon, if the liquid moves into the ergoregion, since ripplons cannot excape from the ergoregion (if the non-relativistic k^4 corrections to the spectrum are ignored). Correspondingly, if the liquid moves from the ergoregion, the boundary of the ergoregion represents the white hole horizon.

The discussed instability of the flow towards formation of ripplons in the supercritical region does not depend on whether the horizon is of a black hole or of a white hole. This mechanism also does not resolve between the ergosurface and horizon. The instability comes from the interaction with the fixed reference frame and occurs in the region where the energy of the critical fluctuation is negative in this frame. Such kind of instability is also called the Miles instability [8]. In principle, Miles instability may take place behind the horizon of the atsronomical black holes if there exists the fundamental reference frame related for example with Planck physics [5, 8]. It may lead to the decay of the black hole much faster than the decay due to Hawking radiation.

4. White-hole horizon in hydraulic jump. The situation with a white hole horizon is achieved in the so-called hydraulic jump first discussed by Rayleigh in terms of the shock wave [9]. The circular hydraulic jump occurs when the vertical jet of liquid falls on a flat horizontal surface. The flow of the liquid at the surface exhibits a ring discontinuity at a certain distance

$r = R$ from the jet (observation of the non-circular hydraulic jumps with sharp corners has been reported in Ref. [10]). At $r = R$ there is an abrupt increase in the depth h of the liquid (typically by order of magnitude) and correspondingly a decrease in the radial velocity of the liquid. The velocity of the liquid in the interior region ($r < R$) exceeds the speed of 'light' for ripplons $v > c = \sqrt{hg}$, while outside the hydraulic jump ($r > R$) one has $v < c = \sqrt{hg}$. Since the velocity flow is radial and outward, the interior region imitates the 'white-hole' region. The interval of the 2+1 dimensional effective space-time in which the "relativistic" ripplons "live" is

$$ds^2 = -c^2 dt^2 + (dr - v(r)dt)^2 + r^2 d\phi^2. \qquad (11)$$

The similar 3+1 dimensional space-time in general relativity, the so-called Painlevé-Gullstrand metric [11]

$$ds^2 = -c^2 dt^2 + (dr - v(r)dt)^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \quad v^2(r) = \frac{2GM}{r}, \qquad (12)$$

is popular now in the black hole physics (see Refs. [12] and references therein).

In general relativity the metric is continuous across the horizon. In our case there is a real physical singularity at the white-hole horizon – the jump in the effective metric (6). However, the discussed mechanism of the Miles instability in the ergoregion (or behind the horizon) does not depend on whether the horizon/ergosurface is smooth or singular.

A similar condensed matter analog of the black-hole horizon with the physical singularity at the horizon has been also discussed by Vachaspati [13]. At the boundary between two superfluids the speed of sound (and thus the acoustic metric) has a jump, $c_1 \neq c_2$. The acoustic black or white horizon occurs if the superfluid velocity of the flow through the phase boundary is subsonic in one of the superfluids but supersonic in the other one, $c_1 < v < c_2$.

5. Hydraulic jump in superfluids. The analogy between the instability of the surface inside the hydraulic jump and the instability of the vacuum behind the horizon can be useful only if the liquid simulates the quantum vacuum. For that, the liquid must be quantum, and its flow should not exhibit any friction in the absence of a horizon. That is why the full analogy could occur if one uses either the flow of quantum liquid with high Reynolds number, or the superfluid liquid which has no viscosity. Quantum liquids such as superfluid or normal ^{3}He and ^{4}He are good candidates.

The first observation of the circular hydraulic jump in superfluid liquid (superfluid ^{4}He) was reported in Ref. [3]. The surface waves generated in the ergoregion (in the region inside the jump) were observed. The critical ripplon appeared to be stationary in the wall frame in agreement with the Miles instability towards ripplon radiation inside the ergoregion discussed in Sec. 3. This is the first experiment, where the analog of the instability of the vacuum inside the horizon has been simulated. The growths of the critical ripplon is saturated due to the non-linear effects, and then the whole pattern remains stationary (though not static). This is different from the case of the instability observed at the interface between ^{3}He-A and ^{3}He-B, where the instability is not saturated and leads to the crucial rearrangement of the vacuum state: Quantized vortices penetrate into the ^{3}He-B side from ^{3}He-A, they partially screen the ^{3}He-B flow and reduce its velocity back below the threshold for the ripplon formation [7].

Under the conditions of experiment [3] the hydraulic jump in superfluid ^{4}He is very similar to that in the normal liquid ^{4}He. The position R of the hydraulic jump as a function of temperature does not experience discontinuity at the superfluid transition. This suggests that quantized vortices are formed, which provide the mutual friction between the superfluid and normal components. As a result even below the λ-point, the liquid moves as a whole though with lower viscosity because of the reduced fraction of the normal component.

To avoid the effect of the normal component it would be desirable to reduce the temperature or to conduct similar experiments in a shallow superfluid ^{3}He.

The advantage of superfluid ^{3}He is that, as distinct from the superfluid ^{4}He, vortices are not easily formed there: the energy barrier for vortex nucleation in ^{3}He-B is about 10^6 times bigger than temperature [14]. In addition, in superfluid ^{3}He the normal component of the liquid is very viscous compared to that in superfluid ^{4}He. In the normal state the kinematic viscosity is $\nu \sim 10^{-4}\,\mathrm{cm}^2/\mathrm{s}$ in liquid ^{4}He, and $\nu \sim 1\,\mathrm{cm}^2/\mathrm{s}$ in liquid ^{3}He. That is why in many practical arrangements the normal component in superfluid ^{3}He remains at rest with respect to the reference frame of the wall and thus does not produce any dissipation if the flow of the superfluid component is sub-critical. One can also exploit thin films of a superfluid liquid, where the normal component is fixed. The ripplons there represent the so-called third sound (recent discussion on the third sound propagating in superfluid ^{3}He films can be found in Ref. [15]). In 1999 Seamus Davis suggested to use the third sound in superfluid ^{3}He for simulation of the horizons [16].

In normal liquids it is the viscosity which determines the position R of the hydraulic jump (see [17, 18]). The open question is what is the dissipation mechanism which determines the position R of the white-hole horizon in a superfluid flow with stationary or absent normal component when its viscosity is effectively switched off. Since there is no dissipation of the superfluid flow if its velocity is below v_L, one may expect that the same mechanism, which is responsible for dissipation in the presence of the horizon, also determines the position R of the horizon. If so, the measurement of R as function of parameters of the system will give the information on various mechanisms of decay of white hole. If the Miles vacuum instability towards ripplon radiation inside the horizon is saturated as in experiment [3], the other mechanisms will intervene such as the black-hole laser [19], and even the quantum mechanical Hawking radiation of ripplons. The latter should be enhanced at the sharp discontinuous horizon of the hydraulic jump and maybe near the sharp corners of the non-circular (polygonal) hydraulic jump observed in Ref. [10].

It is also unclear whether it is possible to approach the limit of a smooth horizon, without the shock wave of the hydraulic jump; and whether it is possible to construct the inward flow of the liquid which would serve as analog of the black hole horizon.

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