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Dephasing of Josephson qubits close to optimal points

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Decoherence of Josephson qubits can be substantially reduced by tuning their parameters to optimal operation points, with only quadratic coupling to fluctuations. We analyze dephasing due to $1/f$ noise for a two-level system, detuned from an optimal point, i.e., the crossover to the linear-coupling regime, both for free induction decay and for spin-echo experiments. Influence of several noise sources is also discussed.

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Superconducting nanocircuits are promising candidates for implementation of quantum-coherent two-level systems, building blocks of prospective quantum information processing devices [1, 2]. At present, coherence of these circuits is limited mostly by low-frequency noise, often with $1/f$ power spectrum, such as background-charge fluctuations, variations of magnetic flux in superconducting loops, and of the critical current of Josephson junctions [3 – 5].

An efficient method of improving the coherence properties of qubits, the "optimal point strategy", was suggested in [6]: the bias of a charge-phase qubit was tuned to an operation point where the linear coupling to the noise sources vanishes. As a result, the coherence time was extended by 2–3 orders of magnitude compared to earlier experiments. This approach was later applied to reduce the influence of bias-current fluctuations on a flux qubit [5]. This method may be used in combination with the spin-echo-type techniques [3, 5, 7], inherited from NMR.

The long coherence time in these devices allows to generate long-lived quantum-coherent oscillations and to study their decay laws in detail. Thus solid-state qubits may be used as a unique tool to gather information about the properties of the noise [8, 9]. Such studies [3, 5, 10] allow one to investigate the origin and the microscopic mechanism of the noise and to design devices with even better coherence, necessary for large-scale quantum circuits.

Analysis of decoherence at and close to an optimal point is important for superconducting quantum bits. Comparison of dephasing at an optimal point and away provides additional information on the nature and statistical properties of the fluctuations (cf. Ref. [3]). Furthermore, it is relevant since a two-level system may be tuned away from degeneracy for quantum manipulations or readout. In other words, here we analyze the crossover between the optimal bias conditions and the linear-coupling regime far away from degeneracy.

Recent experiments suggest that the low-frequency noise in Josephson circuits is produced by collections of bistable fluctuating systems, a well-known model of the flicker noise [11]. For large collections with a sufficiently regular distribution of parameters, one expects the noise to obey Gaussian statistics due to the central limit theorem. For small number of fluctuators or singular distributions, non-Gaussian effects may strongly influence decoherence [12, 13].

Dephasing of qubits by long-correlated Gaussian noise at optimal points was studied earlier for $1/f$ noise spectrum in the cases of free-induction decay [14] and echo [15] experiments (cf. also Ref. [16]). Here we analyze decoherence in a system subject to Gaussian $1/f$ noise, from one or several independent sources, in the vicinity of an optimal point, such that both quadratic and linear coupling terms are relevant.

We express the dephasing law of a system near an optimal point in terms of the decay law precisely at the optimal point. We show that the ratio of these two dephasing laws is dominated by the low-frequency noise. This allows us to find the relevant laws for free-induction and spin-echo decay. We also generalize the analysis to the case of several noise sources.

Below we assume that the infrared cutoff frequency of $1/f$ noise is the lowest frequency scale in the problem, and in particular, is much lower than $1/t$ for relevant times. In our analysis we exploit this fact shifting, under proper conditions, the spectral weights of the fluctuations between zero frequency and $\omega_{\rm ir}$.

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We consider a two-level system (spin-1/2) with the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\left[\varepsilon_0 + \Delta\varepsilon(X(t))\right]\hat{\sigma}_z + \mathcal{H}_{\mathrm{bath}}, \qquad (1)$$

which describes coupling to a noise source via a fluctuating quantity $X(t)$. In turn, its dynamics is governed by the Hamiltonian of the noise source, $\mathcal{H}_{\mathrm{bath}}$. Physically, $X(t)$ may represent, e.g., gate-voltage or magnetic-flux fluctuations in a Josephson qubit. We assume the $1/f$ spectrum of its fluctuations in the relevant frequency range.

Eq. (1) presents the Hamiltonian in the eigenbasis of the non-perturbed ($\Delta\varepsilon = 0$) part. Only the fluctuations of the level splitting, i.e., the diagonal terms, are accounted for. Note that as far as the influence of the low-frequency noise ($\omega \ll \varepsilon_0$) on the qubit is concerned, the general case reduces to Eq. (1) in the adiabatic approximation [17].

To characterize decoherence, we analyze the decay of the off-diagonal entry of the qubit's density matrix in the eigenbasis. Averaging over the noise realizations, we find

$$\begin{gathered}\langle\widehat{\sigma}_-(t)\rangle \equiv \mathrm{Tr}\,(\widehat{\sigma}_-\widehat{\rho}(t)) = \langle\widehat{\sigma}_-(0)\rangle e^{it\varepsilon_0}P(t),\\ P(t) = \langle \mathrm{T}\, e^{i\int_0^t g(t)\Delta\varepsilon(X(t))dt}\rangle,\end{gathered} \qquad (2)$$

where $g(t) \equiv 1$ for the free induction decay. More generally, the function $g(t)$ accounts for modulations of the qubit-noise coupling. For example, in echo-type experiments $|g(t)| = 1$ and the sign of $g(t)$ is reversed every time when a π-pulse is applied.

In the vicinity of an optimal point, keeping the leading-order terms in the expansion of $\Delta\varepsilon$ in X, we find

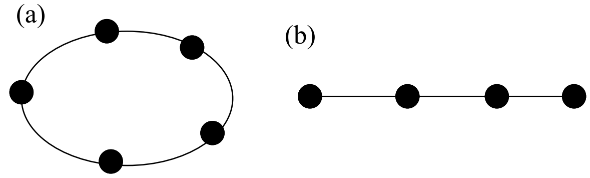
$$\Delta\varepsilon\,(X(t)) = \lambda\,(X(t) + D/2)^2\,, \qquad (3)$$

where we have combined the linear and quadratic terms to form a full square, by transferring a constant from ε_0 for convenience. Here the constant D characterizes the offset from the optimal point and vanishes at the degeneracy.

Dependence on the shift from the optimal point. One can represent the averaging in Eq. (2) via a Gaussian functional integral [15] over $X(t)$, of the type $\int \exp\{-Y^TAY + D(\xi^TY + Y^T\xi) + cD^2\}dY$. This implies a relation between the decay law of coherence in the vicinity of the optimal point, $P(t)$, and the dephasing law $P_0(t)$ at the optimal point (i.e., for $D = 0$):

$$P(t) = P_0(t)e^{-\lambda D^2 f(t)} \qquad (4)$$

with a D-independent function $f(t)$. Naturally, this function depends on $g(t)$ and on the noise power $S_X(\omega)$. Eq. (4) also follows from the diagrammatic analysis [14, 17]: circular diagrams, considered in Ref. [14], contribute to $P_0(t)$, and (the logarithm of) the second term is a combination of linear diagrams, with two linear-coupling vertices at the ends (Figure), thus leading to the D^2 dependence. Below we find $f(t)$ in various situations. Note that in the far-detuned limit $D \to \infty$ only the short-time behavior of $f(t)$ is relevant for the dephasing law.



Diagrams that contribute to the dephasing at an optimal point (a) and close to it (b)

Reduction to the dephasing law at the optimal point. Let us formally average the last equation over a Gaussian-distributed D with dispersion σ^2. Clearly, the average of the dephasing law (4) over this distribution coincides with the dephasing induced by fluctuations with the modified noise power, $\tilde{S}(\omega) = S(\omega) + \frac{\pi}{2}\sigma^2\delta(\omega)$.

Physically, at the time scale t the low-frequency part of $X(t)$ (i.e., the contribution of the frequencies $\omega \ll 1/t$) behaves as a static ($\omega = 0$) random Gaussian quantity, and the dephasing law depends only on its dispersion, i.e., the low-frequency weight $\int d\omega S(\omega)$. Technically, this follows, for instance, from the diagrammatic analysis, similar to that in Ref. [14]. In particular, the addition of a new component at zero frequency is equivalent to the renormalization $\omega_{\mathrm{ir}} \to \omega'_{\mathrm{ir}}$:

$$\lambda\frac{\sigma^2}{4} + \frac{\Gamma}{\pi}|\ln(t\omega_{\mathrm{ir}})| = \frac{\Gamma}{\pi}|\ln(t\omega'_{\mathrm{ir}})|, \qquad (5)$$

where $\Gamma = \lambda X_f^2$, and X_f characterizes the strength of the $1/f$ noise,

$$S(\omega) = X_f^2/|\,\omega\,|. \qquad (6)$$

Thus after averaging (4) over D, one obtains

$$P_0(t, \omega_{\mathrm{ir}}e^{-\frac{\pi\lambda}{4\Gamma}\sigma^2}) = \frac{P_0(t,\omega_{\mathrm{ir}})}{(1 + 2\lambda\sigma^2 f(t))^{1/2}}. \qquad (7)$$

Comparing the subleading terms in the expansion in σ^2 at $\sigma \to 0$, we find [2)]

$$f(t) = \frac{\pi}{4\Gamma} \frac{\partial \ln P_0(t)}{\partial \ln \omega_{\rm ir}}. \quad (8)$$

Taking into account the dimensions of time t, noise power Γ, infrared $\omega_{\rm ir}$ and ultraviolet $\omega_{\rm c}$ cutoff frequencies, one can rewrite Eq. (7) as

$$f(t) = \frac{\pi}{4\Gamma}\left(\frac{\partial \ln P_0(t)}{\partial \ln t} - \frac{\partial \ln P_0(t)}{\partial \ln \Gamma} - \frac{\partial \ln P_0(t)}{\partial \ln \omega_{\rm c}}\right). \quad (9)$$

Eqs. (4) and (8) allow one to find the dephasing close to an optimal point in terms of the dephasing law precisely at this optimal point.

Free induction decay, high ultraviolet cutoff. Short-time asymptotics of $P_0(t)$ were found earlier [14, 15]. For example, in the case of free induction decay and high ultraviolet cutoff, $\omega_{\rm c} t \gg 1$, one immediately finds from Eqs. (4), (8) and the results of Ref. [14] that

$$P(t) = \left(1 - \left(\frac{2}{\pi} i\Gamma t \ln \frac{1}{\omega_{\rm ir} t}\right)\right)^{-1/2} \cdot e^{-\lambda D^2 f(t)}, \quad (10)$$

$$f(t) = \frac{it}{4} \frac{1}{\frac{2}{\pi} i\Gamma t \ln \frac{1}{\omega_{\rm ir} t} - 1}. \quad (11)$$

These results are based on the short-time behavior of $P_0(t)$ at $\Gamma t \ll 1$ (and $|\ln(\omega_{\rm ir} t)| \gg 1$) [14]. They are dominated by the contribution of the low-frequency fluctuations ($|\omega| \ll 1/t$). We show below that this contribution dominates at longer times as well; thus Eq. (11) (but not Eq. (10)) applies at longer times too.

Indeed, we have seen that $f(t)$ can be found from the sensitivity of $P_0(t)$ to $\omega_{\rm ir}$. Clearly, the contribution of high frequencies $\omega \gtrsim 1/t$ is insensitive to the infrared cutoff. Before providing an accurate evaluation of $f(t)$ let us remark that its low-frequency contribution can be estimated by treating the low-frequency part as static noise with the same dispersion, $\sigma^2 = \frac{X_f^2}{\pi} \ln \frac{1}{\omega_{\rm ir} t}$. Then, averaging the phase factor $\exp(i\lambda(X + D/2)^2)$ over the Gaussian distribution $\propto \exp(-X^2/2\sigma^2)$, we find immediately the results (10) and (11).

To calculate $f(t)$ one has to evaluate the diagrams in Figure (b). In contrast to the circular diagrams in Figure (a), the high-frequency contributions in Figure (b) are suppressed since the incoming frequencies at the ends are zero, and at each vertex the typical frequency change is of order $1/t$. Accurate evaluation demonstrates that this indeed suppresses all contributions apart from (11), dominated by very low frequencies $\omega \ll 1/t$.

We remark that in the far-detuned limit $D^2 \to \infty$ the total decay $P_0(t) \exp[-\lambda D^2 f(t)]$ is dominated by the second factor and by the short-time asymptotics of $f(t)$. In this limit one recovers, as expected, the decay laws for a linearly coupled reservoir.

Free induction decay, low ultraviolet cutoff. For $\omega_{\rm c} t \lesssim 1$, $X(t)$ can be treated as static, and one easily gets

$$P_0(t) = \left(1 - \frac{2}{\pi} it\Gamma \ln \frac{\omega_{\rm c}}{\omega_{\rm ir}}\right)^{-1/2}, \quad (12)$$

$$f(t) = \frac{it/4}{\frac{2}{\pi} it\Gamma \ln \frac{\omega_{\rm c}}{\omega_{\rm ir}} - 1}. \quad (13)$$

Spin echo decay near optimal points. For the case of free induction decay, the short-time asymptotics of $P_0(t)$ gave via Eq. (8) an expression for $f(t)$ applicable at all relevant times. Similarly, one can find $f(t)$ for a system subject to spin-echo pulses, using Eq. (8) and the dephasing laws at the optimal point in the corresponding cases [15] (for one or several π-pulses and depending on the value and shape of the ultraviolet cutoff).

In particular, for $N-1$ spin-echo pulses ($N \geq 2$) and a sufficiently high ultraviolet cutoff, $\omega_{\rm c} \gg N/t$, one finds

$$f(t) = \frac{\pi}{8\Gamma} \frac{\frac{C_N}{N}\left(\frac{2}{\pi}\Gamma t\right)^2}{1 + \frac{C_N}{N}\left(\frac{2}{\pi}\Gamma t\right)^2 \ln \frac{1}{t\omega_{\rm ir}}}, \quad (14)$$

where C_N is a constant of order 1 [15].

However, the derivation of Eq. (8) for spin-echo decay requires stricter conditions: it implies that the dephasing doesn't change significantly if the offset D from the optimal point is substituted by a slow oscillating offset with frequency $\omega_{\rm ir} \ll 1/t$. As a result of such a substitution the term $\lambda \int g(t) D^2 dt \equiv 0$ is substituted by a non-zero quantity. This does not change the result as long as this quantity is much less than $\lambda D^2 f(t)$. An estimate of the inaccuracy of Eq. (8) leads to the constraint $(t\omega_{\rm ir})t \ll N f(t)$ on its applicability to an echo-type experiment considered, which for relevant experimental parameters ($\omega_{\rm ir} \ll \Gamma$) reduces to $(t\omega_{\rm ir})t \ll N/|\Gamma \ln(t\omega_{\rm ir})|$. For time scales of interest in spin-echo experiments this constraint is satisfied.

Several fluctuating parameters. Typically, the behavior of Josephson qubits is controlled by several parameters, exhibiting low-frequency noise. The results

[2)]For an arbitrary (non-$1/f$) spectrum with a sharp low-frequency cutoff at a certain frequency $\omega_{\rm ir}$ one finds $f(t) = \frac{\pi}{4\lambda S(\omega_{\rm ir})} \frac{\partial \ln P_0}{\partial \omega_{\rm ir}}$.

derived above can be generalized to the situation with many sources of $1/f$ noise in the vicinity of their optimal points.

In the simplest case, various noise sources contribute independently:

$$\Delta\varepsilon = \sum_i \lambda_i X_i^2 + \sum_i \lambda_i X_i D_i. \qquad (15)$$

Then the dephasing is a product of the partial contributions, $P(t) = \prod_i P_i(t)$, regardless of the noise spectra.

However, in general the quadratic part of the perturbation near the optimal point contains cross terms, $X_i X_j$. For instance, this is the case for the flux qubit in Ref. [18] and two fluxes as control parameters. If all noise sources have similar spectra (such that $S_i(\omega) = \text{const} \cdot S_j(\omega)$), one can reduce the problem to that for independent sources by simultaneous diagonalization of the quadratic perturbation and the correlation matrix $\langle X_i(t) X_j(t')\rangle$ (cf. Ref. [15]). For instance, this remark applies to several sources of $1/f$ noise with the same infrared cutoff (and, e.g., sufficiently high ultraviolet cutoffs, $\omega_c^i t \gtrsim 1$, which do not influence the dephasing law strongly).

In general, however, different sources of $1/f$ noise may have different infrared cutoffs. Still, this case can be reduced to the problem of a single noise source. Indeed, to treat this case, one can employ the approach used above in the derivation of (5): one adjusts all the infrared cutoff frequencies, ω_{ir}^i, to the same value (still, $\ll 1/t$) by shifting the rest of the low-frequency spectral weight to zero frequency. As a result, one arrives at the situation with several noise sources with the same spectra (up to an overall factor) and a static random field. At this point, one can average first over the dynamical fluctuations by re-diagonalization as above, and then over the static noise. Notice, that fluctuations with a low ultraviolet cutoff, $\ll 1/t$, may be treated as low-frequency fluctuations in the same spirit.

In particular, the dephasing for a linear combination $X(t) = \sum \alpha_i X_i(t)$ of $1/f$ noise sources with high ultraviolet cutoffs can be obtained following the procedure described. One finds that the dephasing in this case coincides with the dephasing due to a single source of $1/f$ noise with the power $\Gamma = \sum |\alpha_i|^2 \Gamma_i$ and the infrared cutoff

$$\omega_{\text{ir}} = \Pi_i \left(\omega_{\text{ir}}^i\right)^{\alpha_i^2 \Gamma_i/\Gamma}. \qquad (16)$$

In conclusion, we considered the dephasing of a two-level system close to an optimal operation point, in the crossover regime between the quadratic and the linear coupling to the noise source. We found the dependence on the shift from the optimal point, by relating the dephasing law to that at the optimal point with only quadratic coupling. In the case of $1/f$ noise, we present an explicit expression for the ratio of two dephasing laws. Further, we analyzed the influence of several noise sources and demonstrated, how the problem can be reduced to the case of a single source.

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