Energy spectra of quantum turbulence in He II and ^{3}He-B – a unified view

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Submitted 29 December 2005

Quantum turbulence in superfluid He II and in ^{3}He–B that can be regarded as nearly isothermal, isotropic and homogeneous is discussed within the two–fluid model. A general form of the 3D energy spectrum is proposed: at large length scales, where normal and superfluid eddies are locked together by the mutual friction force, the energy spectrum is essentially classical and includes an inertial range of a Kolmogorov K62-form. With increasing wavenumber, k, the normal fluid part of the spectrum terminates due to finite viscosity while the superfluid part of the spectral energy density changes towards k^{-3} and then back into Kolmogorov-like $k^{-5/3}$ again. Agreement with computer simulations and experiments is claimed if account is taken of the turbulent box size and of the energy decay rate.

PACS: 47.27.Ak, 67.40.Vs, 67.57.De

We consider a classically generated turbulence in quantum liquids He II and ^{3}He–B that can be thought of as isothermal, homogeneous and isotropic (HIT). We use the fluid dynamical approach and discuss forms of the 3D energy spectra in HIT in three steps. First, we comment on classical high Reynolds number (Re) energy spectra relevant to these liquids (i.e., He I and normal liquid ^{3}He) above their superfluid transition temperature. Then we consider the limit of zero temperature and, finally, the form of the 3D energy spectrum for HIT at finite temperature in He II and ^{3}He-B will be proposed and justified. We stress that our approach does not directly apply to quantum turbulence generated in superfluid helium by the thermal counterflow.

Above the pressure dependent transition temperature ($T_\lambda \approx 2.17$ K for ^{4}He and $T_c \approx 1$ mK for ^{3}He) both ^{4}He and ^{3}He are ordinary viscous fluids that can be described by the Navier-Stokes equations and their turbulent flow is fully classical. From fluid dynamical point of view normal ^{4}He and ^{3}He liquids differ from each other mainly because of their very different values of kinematic viscosity. While liquid ^{4}He above T_λ possesses the lowest kinematic viscosity, ν, of all known fluids, of order $\nu_4 \approx 2 \cdot 10^{-4}$ cm^2/s [1], liquid ^{3}He at millikelvin temperature is a Fermi liquid ($\nu_3 \propto T^{-2}$) with kinematic viscosity exceeding that of air, comparable with olive oil, of order $\nu_3 \approx 1$ cm^2/s [2]. In principle (see later), both these liquids may become turbulent.

In case of classical high Re HIT we have the 3D turbulent spectrum of a usual form. For low wavenumber k the spectral energy density is of the form $\Phi(k) = Ak^m$, where m is 2 or 4 [2)]. On increasing k, $\Phi(k)$ around the energy-containing length scale $\ell_e = 2\pi/k_e$ exhibits a broad maximum and the inertial range follows, of the form $\Phi(k) = C\varepsilon^{2/3}k^{-5/3}(k\ell_e)^\mu$, where $C = 1.62 \pm 0.17$ [3] is the dimensionless 3D Kolmogorov constant, $\varepsilon = -dE/dt$ is the energy decay rate and μ is the intermittency correction exponent. Intermittency will most likely become an important issue in quantum turbulence as it is in classical turbulence today, but for simplicity we shall neglect it at this stage. The energy spectrum will be terminated at high $k_{\rm diss} \approx (\varepsilon/\nu^3)^{1/4}$ by viscous dissipation. Here $\ell_{\rm diss}$ is the Kolmogorov dissipation length scale $\ell_{\rm diss} = 2\pi/k_{\rm diss}$. The shape of a typical 3D HIT spectrum is shown in Figure (top left).

Below the transition temperature both ^{4}He and ^{3}He become superfluid. They can be described in the framework of the two fluid model, assuming them as consisting of two interpenetrating fluids – the inviscid superfluid and the viscous normal fluid. Circulation in the superfluid component is quantized in units of κ ($0.997 \cdot 10^{-3}$ cm^2/s for He II [1]; $0.664 \cdot 10^{-3}$ cm^2/s for ^{3}He–B) [2]; we assume singly quantized vortices and neglect any structure of their cores. Quantized vortices couple the normal fluid and the superfluid velocity fields by mutual friction. Thus from the hydrodynamical point of view there is hardly any essential difference between He II and ^{3}He–B – we consider them as mixtures of pure superfluids and normal fluids, with kinematic viscosities that differ by about four orders of magnitude. The classi-

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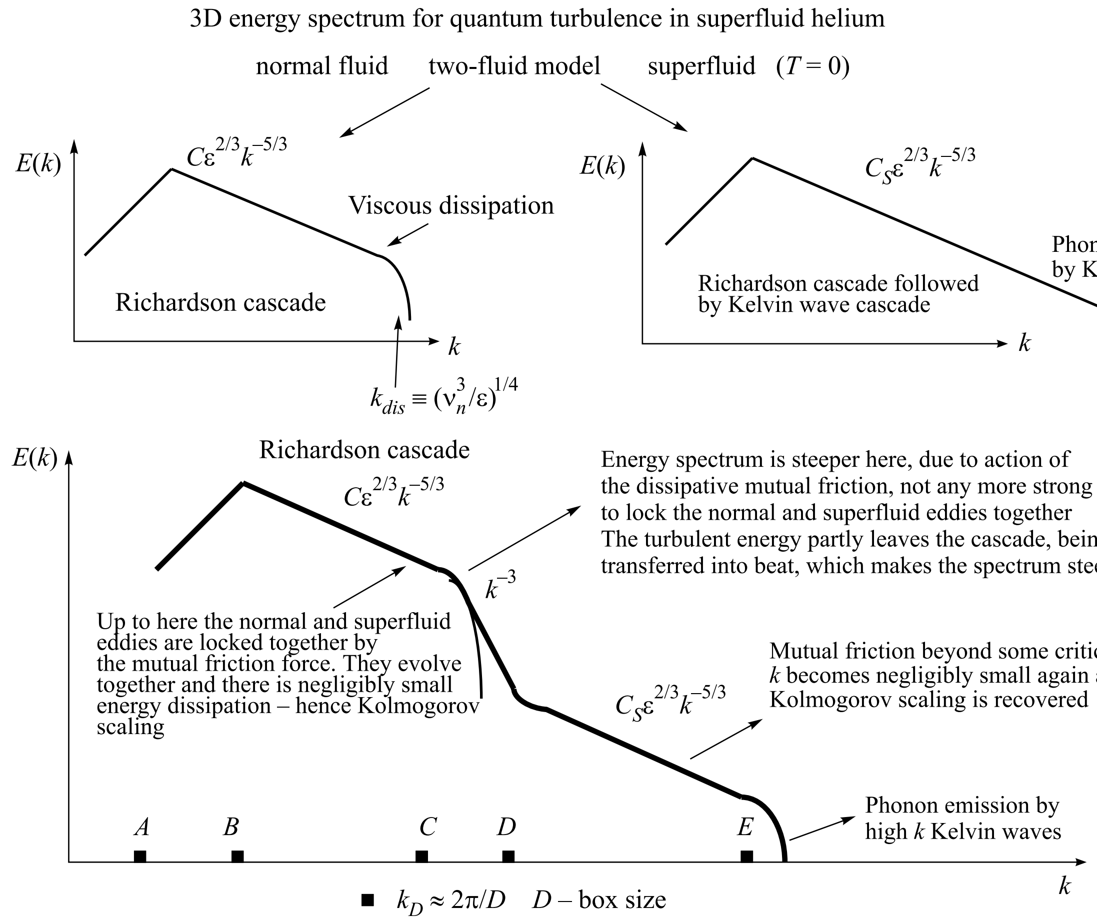
2)Value of m depends on whether Loitsianskii invariant of Saffman invariant are conserved quantities in HIT.



The schematic form of 3D energy spectrum (all axes logarithmic) in a classical viscous fluid (top left) and in pure superfluid (top right) in the $T \to 0$ limit. The bottom figure represents a proposed form of the 3D energy spectrum in highly turbulent helium superfluids at finite temperature

cal high Re 3D energy spectrum outlined above appears due to a Richardson cascade process. The turbulent energy is injected at the rate $\varepsilon = -dE/dt$ into the cascade at a large length scale ℓ_e and propagates down the cascade with essentially no dissipation, until at high k it is viscously dissipated. How is this standard form of the 3D energy spectrum for HIT modified in He II and in ^{3}He–B? Our next step is the $T \to 0$ limit, where we have pure superfluid. The absence of the normal fluid means that the cascade (if it exists) to smaller scales cannot be terminated by viscous damping. We can even wonder whether the quantum turbulence will ever decay? Physical intuition suggests that it must in fact decay, given that vortices represent metastable states of the liquid – and both theory and experiments confirm this.

The theory of quantum turbulence in $T \to 0$ limit and the decay mechanism in this regime (see [4] and references therein) is due especially to Vinen, Barenghi, Brachet, Svistunov, Tsubota and others. The physical picture that is emerging involves a Richardson cascade from the largest length scales to successively smaller scales. At large scales, where big eddies are composed of many aligned quantized vortices, the quantization of circulation is not important and cascade is therefore essentially classical. Then a characteristic quantum wavenumber is reached, defined by dimensional considerations [5] as $k_q \approx (\varepsilon/\nu^3)^{1/4}$. The cascade cannot end here in viscous dissipation, but it is followed by a Kelvin wave cascade. Kelvin waves of successively higher frequency can radiate phonons, thus terminating the cascade. Even if there is no dissipation in some part of the Kelvin wave cascade, the scaling could depend, except on ϵ and k, also on other physical quantities such as the circulation quantum. From dimensional considerations, beyond the corresponding quantum length scale $\ell_q = 2\pi/k_q$, the spectral energy density must be of a form [6] $C_S \epsilon^{2/3} k^{-5/3} f(\epsilon k^{-4} \kappa^{-3})$. The exact scaling here is beyond the scope of this paper. Here we assume the spectral energy density in its usual Kolmogorov K 41 form until it becomes terminated at some high quantum dissipation wavenumber by phonon emission (or, possibly involving the Caroli-Matricon bound states inside vortex cores in ^{3}He–B) [7].

Let us first consider a numerical evidence supporting this picture. The energy spectrum of superfluid turbulence with no normal-fluid component under the vortex filament formulation was obtained by Araki, Tsubota and Nemirovskii [8], starting from a Taylor-Green vortex. The spectral energy density was consistent with the Kolmogorov law, athough the range of the wave numbers was not very wide.

Decaying turbulence originating from the Taylor-Green flow has been numerically studied using the Gross-Pitaevskii equation by Nore, Abid and Brachet [9]. Although the total energy is conserved, the incompressible kinetic energy is changed to the compressible one with emission of sound waves. The spectrum displays the $-5/3$ power on the way of the decay. The situation in the late stage is complicated, however, as the sound waves resulting from vortex reconnections disturb the inertial range and deviate the exponent from $-5/3$. In order to overcome this difficulty, Kobayashi and Tsubota [10] have studied the turbulence within the Gross-Pitaevskii (GP) model by artificially incorporating a small-scale dissipation mechanism. They have solved the GP equation in the wave number space in order to use the fast Fourier transformation and introduced a dissipative term which dissipates the Fourier component of the very high wave number, namely, phonons of short wave length. To solve the GP equation numerically with high accuracy, they have used the Fourier spectral method in space with the periodic boundary condition in a cube. To obtain a vortex tangle, they have started from an initial state where the condensate density is uniform and the phase is random and they have obtained the Kolmogorov spectrum more clearly.

What experimental evidence supports this general picture? In He II, a preliminary experiment in $T \to 0$ limit [11] looks as follows: A fine-mesh grid, stretched to its yield point on a circular holder, is driven on-resonance by an electrostatic field. Because of the high Q of the oscillator, and the fact that it is vibrating in a dissipation-free superfluid, the amplitude rises until the critical velocity for quantum turbulence generation is attained. After the grid has been driven for a few seconds, the drive is switched off. When a pulse of negative ions is passed through the quantum turbulence that has been created, some of the ions get trapped on the vortices, leading to an attenuated signal at the collector. By propagating a sequence of such pulses, the recovery of the signal can be observed. Although direct quantitative comparison of the results of this experiment with the emerging theoretical models is difficult, it clearly illustrates the decay of quantum turbulence on a timescale of seconds.

Bradley et al. [12] generated the quantum turbulence in $T \to 0$ limit in ^{3}He–B also by an (electromagnetically) oscillating grid but detected as a decrease in damping sensed by the nearby vibrating wire [13]. The underlying physics is based on the Andreev scattering (the incoming quasiparticle is back-scattered as a quasihole and vice versa) of ballistically propagating quasiparticles. The scattering appears thanks to Galilean modification of the quasiparticle dispersion relation due to circulating superflow around the vortex core. The results of these experiments [12] are fully consistent with the existence of the Kolmogorov – like inertial range in pure quantum turbulence in the $T \to 0$ limit. It is therefore plausible to suggest that the inertial range of the energy spectrum in the pure superfluid takes identical form as in any viscous fluid, with the phonon irradiation dissipation mechanism replacing at very high k conventional viscous dissipation.

The next step is considering finite temperatures, i.e., mixing the two fluids together in appropriate proportion [1, 2]. We propose a general form of the 3D energy spectrum for quantum HIT in He II and in ^{3}He–B at finite temperature as schematically shown in the bottom part of Figure. Let us explain and justify it.

If quantized vortices are present, normal and superfluid velocity fields are coupled together by the mutual friction force (proportional to the difference in the velocity field), acting at all length scales. It has been shown by Vinen [7] that assuming the normal fluid at rest the superfluid eddies of all sizes would decay exponentially, with some characteristic decay time τ_{mf}. To appreciate the role of mutual friction in turbulent flow, τ_{mf} has to be compared with the eddy turnover time, τ_{to}, at any particular length scale. It is clear that if $\tau_{mf} \ll \tau_{to}$, superfluid and normal eddies are basically locked together. This means that the superfluid and normal fluid velocity fields are (almost) identical and therefore no dissipation occurs due to mutual friction. This is the situation at large scales. So the big normal and superfluid eddies undergo the Richardson cascade together, forming the Kolmogorov inertial range of the form $C\varepsilon^{2/3}k^{-5/3}$, so that the low k parts of the normal and superfluid energy spectra are essentially identical.

On increasing k (approaching either k_q or k_{diss}, whichever is smaller) normal and superfluid eddies start to decouple. There are two reasons for decoupling: (i) finite normal fluid viscosity that acts when approaching the dissipation length scale in the normal fluid [6], and (ii) quantized circulation in superfluid that prevents matching normal and superfluid eddies when approaching the quantum scale [5]. Both these reasons lead to a steeper superfluid energy spectrum, as the energy partly leaks out from the superfluid cascade. We shall see later that there are experimental reasons to expect that the scaling approaches k^{-3}, as shown in Figure.

The subsequent part of the superfluid energy spectrum can be deduced from theory, based on the coarse-grained hydrodynamic equation obtained by Volovik [14, 15] from the Euler equation after averaging over vortex lines written in the frame of reference of the normal fluid:

$$\partial \mathbf{v}/\partial t + \nabla\mu = \mathbf{v}\times\boldsymbol{\omega} + q\hat{\boldsymbol{\omega}}\times(\boldsymbol{\omega}\times\mathbf{v}). \qquad (1)$$

Here $\mathbf{v}$ is the superfluid velocity, $\boldsymbol{\omega}$ denotes the course-grained vorticity and $\hat{\boldsymbol{\omega}}$ is a unit vector in the direction of $\boldsymbol{\omega}$. As was first emphasized by Finne et al. [15], Eq. 1 has a very remarkable property which makes it distinct from the ordinary Navier-Stokes equation where the relative importance of the inertial and dissipative terms is given by the Reynolds number, which in turn depends on the geometry of the particular flow under study. Here the role of the effective Reynolds number is played by the parameter $\mathrm{Re}_{\mathrm{eff}} = q^{-1} = (1+\alpha')/\alpha$ which via the mutual friction parameters α and α' depends on temperature but not on geometry. A wide range of q values is easily experimentally achievable; with q increasing with temperature in both He II [1] and ^{3}He-B [16]. For $T \to 0$ Eq. (1) becomes the Euler equation with no dissipation term, as mutual friction cannot operate in the absence of the normal fluid.

L'vov, Nazarenko and Volovik [17] and independently Vinen [7] have shown that assuming the normal fluid at rest owing to the action of mutual friction, there is strong damping of large eddies, with the result that at low wave numbers the energy spectrum rolls off more rapidly, approaching k^{-3}. This theory is applicable for $k > k_{\mathrm{diss}}$ – normal fluid does not move any more at these scales. Beyond a certain critical value k_{cr}, however, the superfluid energy spectrum gradually recovers its usual Kolmogorov form. Here the normal and superfluid velocity fields are again nearly independent, as the action of the mutual friction force at these small length scales is negligibly small. One can also understand this fact as an opposite sign of the above inequality $\tau_{mf} \gg \tau_{to}$ – mutual friction cannot appreciably affect motion of such small eddies. Only superfluid eddies exist at these small length scales, the normal fluid being at rest due to finite kinematic viscosity. The superfluid Kolmogorov-like cascade exists here on its own, on a background of the stationary (at these scales) normal fluid, until it is terminated by phonon irradiation (this dissipation channel is not contained in Eq. (1)).

Is such a general form of the energy spectrum in accordance with the available experimental results? In answering this question we must take into account the characteristic power typically applied into liquid helium flow in these experiments – it uniquely determines the characteristic length scales ℓ_{diss} and ℓ_q discussed above. Another very important length scale is the size of the turbulent box, D, as eddies larger than the box size cannot exist. The experimentally observed spectrum appears very different if the size of the box intervenes, as marked by letters A–E in Figure.

Let us start with He II, where in relevant experiments the typical power applied to 1 liter of liquid of density 145 kg/m^3 is about 1 W. Here ℓ_{diss} and ℓ_q at temperatures where most experiments have been performed are roughly equal [3)], both of order 10 μm. Using the known relationship between the energy decay rate and vorticity $\varepsilon = \nu_{\mathrm{eff}}\omega^2 \approx \nu_{\mathrm{eff}}(\kappa L)^2$, where ν_{eff} is the temperature dependent effective kinematic viscosity, we have a typical vortex line density of order $L \simeq 10^{11}$ m^{-2}. This corresponds to a typical vortex line spacing, $1/\sqrt{L}$, of a few μm, in agreement with the quantum length marking a quantum unit of circulation. In a turbulent box of cm size we therefore have the possibility of observing the entire spectrum as shown in Figure (i.e, starting from wavenumber marked as A). With ℓ_{diss} and ℓ_q so small it is, however, a challenging experimental task yet to be accomplished. So far the only direct observation of the steady state energy spectrum in He II is that of Mauer and Tabeling [18]. By monitoring the pressure fluctuations using a Pitot tube like sensor they indeed observed the classical-like part of the inertial range both above and below the lambda point, including the intermittency corrections. All the measured spectra appear identical, regardless that the lowest temperature was 1.4 K, at which there is only a few per cent of the normal fluid. In this experiment, due to finite size of the sensor (about 2 mm), any expected changes of the roll-off exponent around ℓ_{diss} and ℓ_q have clearly been out of reach.

Most experiments on quantum turbulence in He II has been performed using the second sound attenuation method. This technique, based on collective physical phenomenon, is extremely sensitive in providing spatially averaged information, namely the projection of the vortex line density to a plane perpendicular to the direction along which the second sound propagates [19]. In the towed grid experiments, up to six orders of magnitude of the decaying vortex line density could be observed over several decades of time, leading to the identification of four different decay regimes [20], as predicted by the spectral decay model [21]. The first three of them, characterized by the power law decay of the vortex line

[3)] While ℓ_q is temperature independent, ℓ_{diss} via ν_n depends on temperature [1]; ℓ_{diss} is slightly smaller than ℓ_q below the lambda point, they become equal at about 1.5 K below which ℓ_{diss} grows, becoming about 100 times larger than ℓ_q at 1 K.

density with exponents $-11/10$, $-5/6$, and $-3/2$, are fully consistent with the classical form of the HIT energy spectrum as described in the beginning of this paper. Decaying He II turbulence serves as a microscope magnifying all of the relevant scales discussed above – they all gradually grow until they eventually become saturated by the size of the experimental channel. Therefore the late decay provides access to small scales that would otherwise be beyond reach using existing up-to-date local probes. The late exponential decay of the vortex line density, which follows the three power-law regimes is consistent with the spectral energy density with the roll off exponent -3 that follows the classical-like $k^{-5/3}$ inertial range in the superfluid component (beyond C in Figure), as discussed in [5].

In ^{3}He–B the situation is different in that the sub-millikelvin temperature only allows applying about 1 nW of power into 1 cm^3 of liquid of density about 100 kg/m^3. Contrary to He II, here $\ell_{\rm diss}$ and ℓ_q are very different, due to the much larger value of the normal fluid viscosity of order $\nu_3 \approx 1$ cm^2/s. For a typical ^{3}He–B experiment on quantum turbulence with $\epsilon \approx 10^{-5}$ m^2/s^3 we thus have $\ell_{\rm diss} \simeq 10$ cm, while $\ell_q \simeq 0.5$ mm. In a turbulent box of typical size of order 1 cm or even less the ^{3}He–B normal fluid cannot become turbulent and can be regarded as always stationary, as it was indeed claimed in the Helsinki experiments [15]. It is interesting to apply Vinen's criterion for the biggest possible size of a superfluid eddy in the stationary normal fluid $R_{\rm max} \approx \epsilon^{1/2}/(\alpha\kappa L)^{3/2}$ [7], where α is the dissipative mutual friction coefficient [16]; $\alpha \leq 1$ if superfluid turbulence exists. Bigger superfluid eddies cannot exist, as they would be destroyed by mutual friction within one turnover time. Assuming that ϵ does not exceed the typical value given above, we arrive at $R_{\rm max} \approx 2$ cm at $T/T_c \approx 0.6$, where the temperature-dependent transition to superfluid turbulence in a cylindrical cell of 0.6 cm in diameter was observed [15].

Let us stress that our considerations are applicable only to flows that can be regarded, at least approximately, as homogeneous, isotropic and isothermal. They do not directly apply to quantum turbulence generated thermally – the thermal counterflow tries to tear the normal and superfluid eddies apart and the classical-like part of the energy spectrum cannot exist. Counterflow turbulence, although its decay in some cases displays classical features [22, 23], is beyond the scope of this paper and will be discussed elsewhere.

To summarize, within the frame of the two-fluid model we have proposed the form of the 3D energy spectrum for isothermal homogeneous and isotropic quantum turbulence in superfluid He II and ^{3}He–B as schematically drawn in Figure. It is in agreement with theoretical models and available experiments on quantum turbulence in He II and ^{3}He–B if the turbulent box size and the energy decay rate are taken into account.

Discussions with many colleagues, especially with D. Charalambous, V.B. Eltsov, S.N. Fisher, M. Krusius, V.S. L'vov, P.V.E. McClintock, J.J. Niemela, G.R. Pickett, K.R. Sreenivasan, M. Tsubota, W.F. Vinen and G.E. Volovik are warmly acknowledged. This research is supported by the Institutional Research Plan AVOZ10100520 and by GAČR under 202/05/0218.

1. R. J. Donnelly and C. F. Barenghi, J. Phys. Chem. Data **27**, 1217 (1998).
2. D. Volhard and P. Völfle, *The superfluid phases of Helium 3*, Taylor and Francis, London, 1990.
3. K. R. Sreenivasan, Phys. Fluids **7**, 2778 (1995).
4. W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. **128**, 167 (2002).
5. L. Skrbek, J. J. Niemela, and K. R. Sreenivasan, Phys. Rev. E **64**, 067301 (2001).
6. W. F. Vinen, Phys. Rev. B **61**, 1410 (2000).
7. W. F. Vinen, Phys. Rev. B **71**, 024513 (2005).
8. T. Araki, M. Tsubota, and S. K. Nemirovskii, Phys. Rev. Lett. **89**, 145301 (2002).
9. C. Nore, M. Abid, and M.E. Brachet, Phys. Fluids **9**, 2644 (1997).
10. M. Kobayashi and M. Tsubota, Phys. Rev. Lett. **94**, 065302 (2005).
11. S. I. Davis, P. C. Hendry, and P. V. E. McClintock, Physica B **280**, 43 (2000).
12. D. I. Bradley et al., *The Decay of Pure Quantum Turbulence in Superfluid ^{3}He-B*, accepted to Phys. Rev. Lett.
13. S. N. Fisher, A. J. Hale, A. M. Guénault, and G. R. Pickett, Phys. Rev. Lett. **86**, 244 (2001).
14. G. E. Volovik, JETP Letters **78**, 553 (2003).
15. A. P. Finne et al., Nature **424**, 1022 (2003).
16. T. D. C. Bevan et al., J. Low Temp. Phys. **109**, 423 (1997).
17. V. S. L'vov, S. V. Nazarenko, and G. E. Volovik, JETP Letters **80**, 479 (2004).
18. J. Maurer and P. Tabeling, Europhys. Lett. **43**, 29 (1998).
19. H. E. Hall and W. F. Vinen, Proc. Roy. Soc. A **238**, 204, 205 (1956); W. F. Vinen, Proc. Roy. Soc. A **240**, 114, 128 (1957); A **242**, 489 (1957).
20. L. Skrbek, J. J. Niemela, and R. J. Donnelly, Phys. Rev. Lett. **85**, 2973 (2000).
21. L. Skrbek and S. R. Stalp, Phys. Fluids **12**, 1997 (2000).
22. L. Skrbek, A. V. Gordeev, and F. Soukup, Phys. Rev. E **67**, 047302 (2003).
23. A. V. Gordeev, T. V. Chagovets, F. Soukup, and L. Skrbek, J. Low Temp. Phys. **138**, 549 (2005).