Differential approximation for Kelvin-wave turbulence

S. Nazarenko

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

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I present a nonlinear differential equation model (DAM) for the spectrum of Kelvin waves on a thin vortex filament. This model preserves the original scaling of the six-wave kinetic equation, its direct and inverse cascade solutions, as well as the thermodynamic equilibrium spectra. Further, I extend DAM to include the effect of sound radiation by Kelvin waves. I show that, because of the phonon radiation, the turbulence spectrum ends at a maximum frequency $\omega^* \sim (\epsilon^3 c_s^{20}/\kappa^{16})^{1/13}$ where ϵ is the total energy injection rate, c_s is the speed of sound and κ is the quantum of circulation.

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1. Kelvulence: cascades and spectra. Kelvin waves propagating on a thin vortex filament were proposed by Svistunov to be a vehicle for the turbulent cascades in superfluids near zero temperature [1]. Presently it is a widely accepted view well supported by the theory and numerical simulations, see e.g. [2 – 6]. I will refer to the state characterised by random nonlinearly interacting Kelvin waves as "kelvulence" (i.e. Kelvin turbulence). Recently, Kozik and Svistunov [4] used the weak turbulence approach to kelvulence and derived a six-wave kinetic equation (KE) for the spectrum of weakly nonlinear Kelvin waves. Based on KE, they derived a spectrum of waveaction that corresponds to the constant Kolmogorov-like cascade of energy from small to large wavenumbers,

$$n_k \sim k^{-17/5}. \quad (1)$$

Because the number of waves in the leading resonant process is even (i.e. [6]), KE conserves not only the total energy but also the total waveaction of the system. The systems with two positive conserved quantities are known in turbulence to possess a dual cascade behavior. For the Kelvin waves, besides the direct energy cascade there also exists an inverse cascade of waveaction, the spectrum for which was recently found by Lebedev [7],

$$n_k \sim k^{-3}. \quad (2)$$

Interestingly, such -3 spectrum was suggested before by Vinen based on a dimensional argument not involving the energy flux [2]. Similar argument in water wave turbulence gives famous Phillips spectrum which is associated with sharp water crests due to wavebreaking which occurs at large excitation levels. By analogy, we could expect that Vinen -3 spectrum should be observed in kelvulence at high excitation levels leading to sharp angles due to reconnections. This view is supported by recently reported numerics by Vinen, Tsubota and Mitani [5], where it was argued that the -3 exponent arises when the vortex line bending angle becomes large (of order one). Kozik and Svistunov [6] later reported a result obtained with a refined numerical method which gave a spectrum closer to the $-17/5$ shape, which is a Kolmogorov-like direct cascade of energy dominating the wavebreaking effects at lower turbulence levels. The inverse cascade spectrum, although it has the same -3 exponent as the Vinen spectrum, is fundamentally different: it corresponds to weak rather than strong turbulence and is more relevant if the main source of the Kelvin waves is at small scales. For example, the smallest scales on the vortex filament could be generated by reconnections via sharp angles produced by these processes.

2. Differential equation model. Differential equation models proved to be very useful for analysis in both weak turbulence [8 – 11] and strong turbulence [12 – 14]. These equations are constructed in such a way that they preserve the main scalings of the original closure (KE in the case of weak turbulence), in particular, its nonlinearity degree with respect to the spectrum and its cascade and thermodynamic solutions. For the Kelvin wave spectrum these requirements yield,

$$\dot{n} = \frac{C}{\kappa^{10}} \omega^{1/2} \frac{\partial^2}{\partial \omega^2} \left(n^6 \omega^{21/2} \frac{\partial^2}{\partial \omega^2} \frac{1}{n} \right), \quad (3)$$

where κ is the vortex line circulation, C is a dimensionless constant and $\omega = \omega(k) = \frac{\kappa}{4\pi} k^2$ is the Kelvin wave frequency (here, we ignore logarithmic factors). This equation preserves the energy

$$E = \int \omega^{1/2} n \, d\omega \quad (4)$$

and the waveaction

$$N = \int \omega^{-1/2} n \, d\omega. \tag{5}$$

It is an easy calculation to check that equation (3) has both the direct cascade solution (1) and the inverse cascade solution (2). It also has the same thermodynamic Rayleigh-Jeans solutions as the original KE,

$$n = \frac{T}{\omega + \mu}. \tag{6}$$

where T and μ are constants having a meaning of temperature and the chemical potential respectively.

3. Kelvulence radiating sound. In contrast with the classical Navier-Stokes flow, there is no viscosity that could dissipate the superfluid turbulent cascade at small scales. At low temperatures, friction with the normal component is also inefficient and the only dissipative process which can absorb the cascade is radiation of sound by moving superfluid vortex filaments [2]. Let introduce the sound dissipation effect into differential equation model (DAM) by using the classical results of Lighthill about sound produced by classical turbulence [15]. Namely, we will use the result that the rate at which the sound energy is generated is proportional to the fourth power of the Mach number $M = v/c_s \sim n^{1/2}/c_s$, where v is characteristic velocity in turbulence and c_s is the speed of sound [1)] Further, we will ignore the logarithmic corrections due to the finite vortex core size a. In the other words, a should not enter the expression explicitly but only implicitly via $c_s \sim \kappa/a$ (which in turn enters only via M). Then the rest of the sound dissipation term can be completed uniquely via the dimensional argument and the result is

$$(\dot{n})_{\text{radiation}} = -\frac{\omega^{9/2} n^2}{\kappa^{1/2} c_s^4}. \tag{7}$$

Let us now examine what effect of the sound radiation on the direct energy cascade. For these purposes we can consider even simpler first order DAM which still describes the direct cascade but ignores the inverse cascade and the thermodynamic solutions. Such DAM, including the sound radiation term, reads:

$$\dot{n} = \frac{C_1}{\kappa^{10}} \omega^{-1/2} \frac{\partial}{\partial \omega} \left(n^5 \omega^{17/2} \right) - \frac{C_2 \omega^{9/2} n^2}{\kappa^{1/2} c_s^4}, \tag{8}$$

[1)] A different (linear) mechanism of sound generation was proposed by Vinen in [16], but it was also shown in this paper that the net sound power generated by this mechanism due to Kelvin wave is zero due to interference effects. Thus we assume that the first nonvanishing contribution comes from the nonlinear terms, same as in the Lighthill's theory, and this dictates M^4 dependence.

where C_1 and C_2 are dimensionless constants. The general stationary solution of equation (8) is

$$n = A \left(1 - B\omega^{13/5}\right)^{1/3} \omega^{-17/10}, \tag{9}$$

where

$$A = (2/C_1)^{1/5} \kappa^{21/10} \epsilon^{1/5}, \tag{10}$$

$$B = \frac{3 \cdot 2^{-3/5}}{13} C_2 \kappa^{16/5} c_s^{-4} \epsilon^{3/5}, \tag{11}$$

and ϵ is the total energy dissipated in the system per unit time per unit length.

In the absence of sound radiation, $B = 0$, we recover the direct cascade spectrum (1). For $B > 0$, the spectrum has the direct cascade shape (1) at low frequencies, $\omega \ll B^{-5/13}$, and it falls to zero at a finite frequency

$$\omega^* \sim B^{-5/13} \sim (\epsilon^3 c_s^{20}/\kappa^{16})^{1/13}. \tag{12}$$

On the other hand, Kelvin waves can only have wavelengths greater than the vortex radius a which in terms of the frequency means $\omega < \omega_c = c_s^2/\kappa$. Thus, expression (9), particularly the finite cut-off at ω^*, will only hold if $\omega^* < \omega_c$ or

$$\epsilon < \kappa c_s^2. \tag{13}$$

This condition can be formulated in terms of the characteristic bending angle of the vortex line α which in the direct cascade state is related to ϵ as

$$\alpha \sim (\epsilon l^2/\kappa^3)^{1/10}, \tag{14}$$

where l is characteristic length of the Kelvin waves. Thus, condition $\omega^* < \omega_c$ becomes

$$\alpha < (l/a)^{1/5}, \tag{15}$$

which always holds for $\alpha \lesssim 1$ because $l > a$. Therefore we conclude that the kelvulence cascade will always decay to zero before reaching the maximal allowed frequency of the propagating Kelvin waves.

4. Conclusions. In this Letter I presented DAM for the Kelvin wave turbulence (kelvulence) consistent with the scalings of, and having the same set of cascade and thermodynamic solutions as, the original six-wave kinetic equation. DAM also exists in reduced versions if some of the flux and thermal solutions are not important for a particular problem and can be ignored (in this Letter I presented three versions given by the fourth, the second and the first order equations respectively).

Based on the Lighthill's theory of sound produced by turbulence, I extended DAM to include the effect of kelvulence dissipation by sound radiation. I obtained a steady state solution of this model corresponding to the direct cascade of wave energy gradually dissipated via radiation and decaying to zero at a finite wavenumber ω^* given by the expression (12). This solution can also be used to predict the spectrum of the radiated sound, because the sound energy is just the same as the energy lost by kelvulence via radiation. One can also extend this study to scattering of sound by turbulence in systems where such sound is generated by externally.

Finally, in Appendix I presented a "warm cascade" solution for kelvulence which is relevant for numerical simulations with a finite cut-off frequency.

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Appendix: Warm cascade solutions

Above, we ignored the "thermal" component in turbulence which is totally justified because this component does not show up when turbulence is dissipated gradually in the Fourier space, as it is done via the sound radiation in our case. On the other hand, it is known in the turbulence theory that sharp dissipation or presence of a cut-off frequency lead to reflection of a large portion of the energy flux from the smallest scale and, consequently, an pile-up of the spectrum near the smallest scale. This "bottleneck" effect can be described in terms of "warm cascade" solutions in which both an energy flux and a thermal components are present. As I showed above, the natural cut-off frequency of Kelvin waves ω_c is not going to be reached by the turbulent cascade because it will always terminate at a frequency $\omega_* < \omega_c$ due to the sound radiation. However, in numerical simulations of kelvulence the cut-off frequency may be less than ω_* due to the limited numerical resolution. Thus, the warm cascade solutions could be relevant for understanding spectra obtained numerically. Such warm cascade solutions were originally obtained for classical Navier-Stokes turbulence in [13] and here we will follow a similar approach to find such states for kelvulence. To describe both the energy cascade and the thermodynamic component (but still ignore the inverse cascade solution) we need to use the second order version of DAM, namely

$$\dot{n} = \frac{C}{\kappa^{10}}\omega^{-1/2}\frac{\partial}{\partial\omega}\left(n^4\omega^{17/2}\frac{\partial(\omega n)}{\partial\omega}\right). \qquad (16)$$

The general steady state solution of this equation is

$$n = \omega^{-1}\left(\frac{20}{7C}\kappa^{21/2}\epsilon\omega^{-7/2} + T^5\right)^{1/5}, \qquad (17)$$

where T is an arbitrary constant having the meaning of "temperature". This is the "warm cascade" solution which has the pure cascade solution and the thermodynamic energy equipartition as two of its limits. Note that the thermodynamic part of this solution describes the spectrum pile-up due to the bottleneck phenomenon in numerical simulations due to presence of a cut-off frequency. The relative strength of the cascade and the thermodynamic components will be determined by the ratio of the incident and the reflected energy fluxes which, in turn, will depend on the particular form of dissipation at small scales chosen in numerical simulations. The goal of the true numerical simulation is to chose the dissipation function in such a way that the bottleneck effect is minimal or absent.

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