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A model of laminated wave turbulence

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A model of laminated wave turbulence is presented. This model consists of two co-existing layers – one with continuous waves' spectra, covered by KAM theory and Kolmogorov-like power spectra, and one with discrete waves' spectra, covered by discrete classes of waves and Clipping method. Some known laboratory experiments and numerical simulations are explained in the frame of this model.

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1. Continuous wave spectra. In [1] Kolmogorov presented energy spectrum of turbulence describing the distribution of the energy among turbulence vortices as function of vortex size and thus founded the field of mathematical analysis of turbulence. Kolmogorov regarded some inertial range of wave numbers, between viscosity and dissipation, and suggested that at this range, turbulence is (1) locally homogeneous (no dependence on position) and (2) locally isotropic (no dependence on direction) which can be summarized as follows: probability distribution for the relative velocities of two particles in the fluid only depends on the distance between particles. Using these suggestions and dimensional analysis, Kolmogorov deduced that energy distribution, called now Kolmogorov´s spectrum, is proportional to $k^{-5/3}$ for wave numbers k. Results of numerical simulations and real experiments carried out to prove this theory are somewhat contradictious. On the one hand, probably the most spectacular example of the validity of Kolmogorov's spectra is provided in [3] where measurements in tidal currents near Seymour Narrows north of Campbell River on Vancouver Island were described and $-5/3$ spectra appeared at the range of 10^4 (energy dissipation at a scale of millimeters and energy input – at 100 m). On the other hand, Kolmogorov´s spectra have been obtained under the assumptions opposite to Kolmogorov's [4] so that exponent $-5/3$ corresponds to both direct and inverse cascades.

With a hope to diminish established unclearness of Kolmogorov's theory in a more simple setting, theory of wave (or weak) turbulence (WT) in the systems with continuous wave spectra has been developed. The problem is regarded in the very general form

$$L(\psi) = -\varepsilon N(\psi), \tag{1}$$

where L and N are linear and nonlinear operators consequently, $0 < \varepsilon \ll 1$ is a parameter of nonlinearity and linear part possesses wave-like solutions of the form

$$\psi(\mathbf{x}) = A(\mathbf{k}, x) \exp i(\mathbf{kx} - \omega(\mathbf{k})t). \tag{2}$$

Choice of ε is defined by specifics of the physical wave system under the study. For instance, for spherical planetary waves ε is usually chosen as the ratio of the particle velocity to the phase velocity while for water waves usually $\varepsilon = a|\mathbf{k}|$, where a is amplitude of a wave and ε characterizes in this case the steepness of the waves. For small enough ε, solutions of Eq.(1) are described at the slow-time scales $T_1 = t/\varepsilon$, $T_2 = t/\varepsilon^2$, $T_3 = t/\varepsilon^3$, ... , etc. by resonantly interacting waves only, i.e. by the waves with wave-vectors satisfying to resonant conditions (for nonlinearity of order n):

$$\omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm ... \pm \omega(\mathbf{k}_{n+1}) = 0, \tag{3.1}$$

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm ... \pm \mathbf{k}_{n+1} = 0 \tag{3.2}$$

so that quadratic nonlinearity corresponds to 3-waves interactions, cubic – to 4-waves interactions, etc. Obviously, in every physical problem the resonances have some nonzero width, i.e. Eq.(3.1) takes form

$$\omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm ... \pm \omega(\mathbf{k}_{n+1}) = \Delta \tag{4}$$

with a small but nonzero discrepancy $0 < \Delta \ll 1$. Taking this into account, nonlinear part of (1) can be rewritten as

$$\Sigma_i \frac{V_i \delta(\mathbf{k}_1 \pm \mathbf{k}_2 \pm ... \pm \mathbf{k}_i)}{\omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm ... \pm \omega(\mathbf{k}_i)}, \tag{5}$$

where δ is a delta-function and each term of this sum with nonzero vertex coefficient V_i corresponds to a specific slow-time scale of wave interactions. The representation (5) is used then for construction of a wave

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kinetic equation, with corresponding vertex coefficients and delta-functions in the under-integral expression.

From mathematical point of view, it is important to establish the **finiteness** of nonlinearity given by (5) in the case when $\Delta \to 0$ because representation (5) becomes meaningless if $\Delta = 0$. This problem – so-called "problem of small denominators" – was solved by KAM theory ([13, 12, 15]) in the following way. For small enough Δ and **sufficiently** irrational dispersion function ω, these wave systems contain an infinite set of invariant tori which carry quasi-periodic motions which in phase space are confined to the tori. Main result of KAM theory is therefore a decomposition of action into disjoint invariant sets, and though it contradicts ergodicity but not very substantially as the size of the system tends to infinity [2]. In particularly, random phase approximation can be assumed and kinetic equations and Kolmogorov-like power spectra k^{γ}, $\gamma < 0$, give then appropriate description of these wave systems at the corresponding time scales. It means that some special set of points in spectral space, corresponding to the Δ-vicinity of exact resonances, has been excluded from consideration in order to obtain wave kinetic equation. We give detailed description of this subset of spectral space in the section 4 of this paper.

2. Discrete wave spectra. There exist a lot of wave phenomena which are due to discreteness of the wave spectra (corresponds to zero- or periodic boundary conditions) and can not be explained in terms of kinetic equations and power energy spectra. To describe these phenomena, WT in the systems with discrete spectra has been developed [10, 11]. It turned out that discrete systems possess some qualitatively new properties: (a) all resonantly interacting waves are divided into disjoint classes, there is no energy flow between different these classes; (b) major part of the waves do not interact; (c) all interactions of a specific wave are confined to some finite domain; (d) number of interacting waves depends on the form of boundary conditions, for the great number of boundary conditions interactions are not possible; (e) all properties (a)-(d) keep true for approximate interactions, i.e. for some small enough discrepancy $0 < \Delta \ll 1$. This fact gave a rise to Clipping method [9] which allows "to clip out" all non-interacting waves from the whole spectra and study only those which do interact, exactly or approximately. Approximate resonances are understood on a discrete lattice, i.e. wave vectors of approximately interacting waves are also integers. The energetic behavior of these systems is described then independently (at each slow-time scale T_j) by a few small systems of ordinary differential equations (SODE) on slowly-changing amplitudes of resonantly interacting waves, i.e. amplitude of linear wave (2) is a function of some T_j depending on the form on nonlinearity N. For instance, for 3-waves interactions major part of these SODE consist of three equations on three (real-valued) amplitudes and can be solved explicitly in terms of elliptic functions on T_1. To compare with WT of the systems with continuous spectra, SODE are to be used instead of kinetic equation and their coupling coefficients – instead of power-law spectra. Notice that though kinetic description does not apply for discrete systems, some of these results are in a sense similar to those of KAM theory, for instance, Theorem on the partition [7] can be regarded as an analog of KAM-Theorem for discrete systems.

3. Transition from discrete to continuous spectra. Now, the standard qualitative model of the WT can be presented as follows: short waves are described by Kolmogorov's energy spectra and kinetic equations, long waves are described by Clipping method and dynamic equations, and somewhere in between a "transition" interval exists that has its own specifics and should be described separately.

We would like to demonstrate some contradictiveness of this qualitative model and begin with two remarks. (1) WT of discrete waves systems has been developed for arbitrary wave numbers which means that transition from finite to infinite domain can be constructed not only in some finite "transition" interval but at the whole infinite range of wave numbers. (2) Transition from discrete to continuous spectrum is often regarded in somewhat over-simplified way: if say, real-valued wave vectors $\mathbf{k} = (k_x, k_y)$ have dispersion function $\omega(k_x, k_y)$ with $k_x, k_y \in \mathbb{R}$, then the same function of integer variables, $\omega(m, n)$ with $m, n \in \mathbb{Z}$, describes corresponding discrete waves. In general, it is not true. We demonstrate it taking barotropic vorticity equation (BVE), also known as Obukhov-Charney-Hasegawa-Mima equation, as our main example motivated by its wide applicability for describing a great number of physically important phenomena in astrophysics, geophysics and plasma physics.

BVE on a sphere has form

$$\frac{\partial \triangle \psi}{\partial t} + 2\frac{\partial \psi}{\partial \lambda} + J(\psi, \triangle \psi) = 0 \qquad (6)$$

with linear waves of the form

$$\psi_{\text{sphere}} = A P_n^m(\sin\phi) \exp i[m\lambda + \omega_{\text{sphere}} t]. \qquad (7)$$

Here ψ is the stream-function; variables t, ϕ and λ physically mean the time, the latitude $(-\pi/2 \le \phi \le \pi/2)$ and the longitude $(0 \le \lambda \le 2\pi)$ respectively; $P_n^m(x)$ is the associated Legendre function of degree n and order m.

The same equation taken on infinite β-plane has linear waves of the form

$$\psi_{\text{plane}} = A \exp i(k_x x + k_y y + \omega_{\text{plane}} t),$$

which means that

$$\omega_{\text{sphere}} = m/[n(n+1)] \quad \text{and} \quad \omega_{\text{plane}} = k_x/(1+k_x^2+k_y^2),$$

here constant multipliers are omitted because they disappear due to homogeneous form of Eq.(3.1). It is easy to see that no wave vectors $\mathbf{k} = (m,n) : m,n \in \mathbb{Z}$ satisfy Eq.(3.1) with ω_{sphere} and with ω_{plane} simultaneously. It means that discrete waves do not have images on infinite plane when such a "naive" transition is regarded.

More intrinsic construction of the transition from spherical to plane planetary waves [6] can be derived in following way. Regarding $m \sim n \gg 1$ and using asymptotic approximation for Legendre functions, one can "convert" (not always but in a bounded latitudinal belt with the width $\sim n^{-1}$) one spherical wave into a linear combination of two plane waves

$$A \exp i(k(\varphi_0)_x x \pm k(\varphi_0)_y y + \omega_{\text{plane}} t),$$

where local wave numbers $k(\varphi_0)_x, k(\varphi_0)_y \in \mathbb{R}$ are functions of the initial spherical wave number m,n and of the so-called interaction latitude φ_0:

$$\cos^2\varphi_0 = $$
$$= \frac{m_1^2(n_2^2+n_3^2-n_1^2)+m_2^2(n_1^2+n_3^2-n_2^2)+m_3^2(n_1^2+n_2^2-n_3^2))}{n_1^2n_2^2+n_1^2n_3^2+n_2^2n_3^2-(n_1^4+n_2^4+n_3^4)/4}.$$

If interaction latitude exists, $0 < \cos^2\varphi_0 < 1$, plane images of spherical waves interact as in classical β-plane approximation. In particularly, this means that (1) transition from a spherical domain to an infinite plane is transition to a one-parametric family of infinite planes, and (2) such a transition is not always possible. A very important fact is that plane wave system keeps memory about spherical interactions: coupling coefficient of the plane images of spherical waves is $\sim k^{3/2}$ and $\sim k^{7/6}$ otherwise and $k = |\mathbf{k}|$.

The same reasoning allows to construct a transition from a square domain (dispersion function being then $\omega_{\text{square}} = 1/\sqrt{m^2+n^2}$) to the infinite β-plane where difference in magnitudes of coupling coefficients is even more substantial: $\sim k^2$ for plane images of the waves from square domain and $\sim k$ otherwise [8]. These results hold for discrete approximate interactions in following way: long-wave part of spectrum is dominated by a few resonantly interacting waves with huge amplitudes while short-wave part of the spectrum consists of many approximately interacting waves with substantially smaller amplitudes. But in any case, coupling coefficients are of order k^γ with $\gamma > 0$ which **apparently** contradicts to the existence of Kolmogorov-like power spectra k^γ with $\gamma < 0$ in the region of short waves.

4. Laminated WT. In order to resolve this apparent contradiction we have go back to the very base of the KAM theory. Its main results are based on the famous Thue theorem, giving low estimate for the distance between any algebraic number α of degree $n > 2$ and a rational number $p/q \in \mathbb{Q}$:

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^{\Delta+1+n/2}}, \quad \forall\varepsilon > 0$$

where $c(\alpha)$ is a constant depending on α and Δ can be arbitrary small. This fact allows to construct KAM tori, with α being a ratio of frequencies of interacting waves, and KAM theorem states then that **almost** all tori are preserved. "Almost" means in particular that tori with rationally related frequencies (corresponds to α being an algebraic number of degree $n = 1$) are explicitly **excluded** from consideration. Since the union of invariant tori has positive Liouville measure and $\mathbb{Q}$ has measure 0, this exclusion is supposed to be not very important.

Coming back to the example of spherical planetary waves, one can see immediately that the ratio of the frequencies for resonantly interacting waves in this case is a rational number, both for exact and approximate interactions. Therefore, these waves are not described by KAM theory. Obviously, the same keeps true for an arbitrary wave system with rational dispersion function ω. Case of planetary waves in a square domain is a bit more complicated. It is proven [11] that exact resonances in this case are described by the wave vectors $\mathbf{k}_i = (m_i, n_i)$ satisfying the (necessary) condition

$$k_i = a_i\sqrt{q}, \;\; a_i \in \mathbb{N} \;\; \forall i = 1,2,3,$$

with **the same** square-free q. This fact allows to construct disjoint classes of resonantly interacting waves and q is called index of the class. Obviously, for the waves belonging to the same class, the ratio of their frequencies is a rational number, $\omega_i/\omega_j \in \mathbb{Q}, \;\; \forall i,j$ and these waves are excluded from KAM theory. Waves, interacting approximately, may have different indices (not necessarily) but the ratio of frequencies ω_i/ω_j is then an algebraic number of degree ≤ 2 due to the form of dispersion function and, therefore, these waves are also excluded from the KAM theory. Similar results can be proven for exact resonances in many wave systems, for instance, for an arbitrary wave system in which dispersion function is a polynomial of finite degree on k with at least one non-zero coefficient in front of an odd degree of k.

Let us summarize the results obtained. Continuous WT (CWT) describes energetic behavior of a wave system for the short-waves' part of spectrum **excluding** nodes of rational lattice thus leaving some gaps in the spectrum which are supposed to be not important in short-waves' part. Discrete WT (DWT) fills these gaps all over the spectrum. In fact we have two layers of turbulence – CWT (layer I) and DWT (layer II), which are mutually complementary and should be regarded simultaneously.

Layer I provides KAM tori and stochastic enough turbulence in short-waves range with Kolmogorov´s spectra in the inertial interval; direct or inverse energy cascades are possible; wave-numbers range of energy pumping influences the results.

Layer II provides a countable number of waves with big amplitudes all over the wave spectrum; some of the waves do not change their energies (non-interacting waves) and others do exchange energy within small independent groups; there is no energy cascade at this layer; results do not depend on the wave-numbers range of energy pumping.

The co-existence of these two layers means in particularly that in the waves' range, classically described by Kolmogorov-like spectra, there exist also a small group of waves from layer II with substantially bigger energies than their "neighbors" from layer I, and this small group generates appearance of some structures. We give here a few examples of known phenomena which can be explained in the same frame of the model of laminated turbulence.

(1) Very clear example of the co-existence of these two layers is given in [16] where turbulence of capillary waves was studied in the frame of simplified dynamical equations for the potential flow of an ideal incompressible fluid. A stationary regime of so-called "frozen turbulence" had been discovered: in small wave-numbers region wave spectrum consists of "several dozens of excited low-number harmonics" which construct "ring structures in the spectrum of surface elevation". The appearance of these structures does not depend on the damping and pumping, and in all computations "the Kolmogorov´s spectrum coexists with the spectrum of another, "frozen" type, concentrated in the region of low wave-numbers and fastly decreasing to large wave-numbers. If the level of nonlinearity is low enough, such "frozen" regimes are dominant" ([16]). Obviously, these ring structures are due to non-interacting waves of layer II and similar structures were also observed in laboratory experiments and identified as such [5]. Notice that as there exists no exact three-wave interactions among capillary waves with $\omega^2 = k^3$ [11], we observe in regime of frozen turbulence of capillary waves only discrete waves with constant amplitudes.

(2) Similar experiments/simulations with, say, four-waves interactions among gravity waves, $\omega^2 = k$, will demonstrate that frozen turbulence partly "thaws out" because also changes in the amplitudes of resonantly interacting discrete waves should be observed (cf. "bursty" spectrum in [14]).

(3) Mesoscopic turbulence [17] (corresponds to "transition" interval mentioned above) discovered in numerical experiments on modelling of turbulence of gravity waves on the surface of deep ideal incompressible fluid gives another example of manifestation of laminated turbulence. It was established that not all waves "have the same rights" and the existence of so-called "elite society of harmonics" has been demonstrated; their number amounts to only 6% of the total number of harmonics being 10^4 but they play the most active role in mesoscopic turbulence. These elite harmonics correspond to exact and approximate resonances of the discrete waves (layer II). The number of these harmonics depends, of course, on the specific wave system. For instance computations with spherical planetary waves in the domain of wave numbers $0 < m, n \le 1000$ (which is far beyond the region of applicability of BVE) shows that total number of harmonics taking part in exact resonant interactions is 22683, i.e. slightly more than 2%. Consideration of approximate interactions of the layer II increases this amount but not substantially.

(4) Zonal extended vertices (flows in latitudinal direction) in the atmosphere can possibly be explained in terms of plane images of spherical waves with coupling coefficient $n^{3/2}$.

At the end of this letter we would like to make one important remark. To describe the short waves of layer II it is necessary to develop fast algorithms of solving Diophantine equations in very big integers (of the order 10^{12} and more). We consider it possible basing on the existence of disjoint classes of waves participating in a single solution. This is our current object of interest.

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