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Differential model for 2D turbulence

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We present a phenomenological model for 2D turbulence in which the energy spectrum obeys a nonlinear fourth-order differential equation. This equation respects the scaling properties of the original Navier-Stokes equations and it has both the $-5/3$ inverse-cascade and the -3 direct-cascade spectra. In addition, our model has Raleigh-Jeans thermodynamic distributions, as exact steady state solutions. We use the model to derive a relation between the direct-cascade and the inverse-cascade Kolmogorov constants which is in a good qualitative agreement with the laboratory and numerical experiments. We discuss a steady state solution where both the enstrophy and the energy cascades are present simultaneously and we discuss it in context of the Nastrom-Gage spectrum observed in atmospheric turbulence. We also consider the effect of the bottom friction onto the cascade solutions, and show that it leads to an additional decrease and finite-wavenumber cutoffs of the respective cascade spectra which agrees with existing experimental and numerical results.

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I. Leith differential model and its relatives. In 1967 paper [1], Leith proposed the following differential equation model as a model of 3D turbulence,

$$\frac{\partial E}{\partial t} = \frac{1}{8}\frac{\partial}{\partial k}\Big[\sqrt{k^{11}E}\,\frac{\partial}{\partial k}\Big(\frac{E}{k^2}\Big)\Big] - \nu k^2 E, \qquad (1)$$

where t is time, k is the absolute value of the wavenumber, ν is the kinematic viscosity and the one-dimensional energy spectrum $E(k,t)$ is normalised so that the kinetic energy density is $\int E\,dk$. The second term on the RHS of Eq. (1) is obvious and describes the linear process of viscous dissipation. The first term contains a nontrivial model of the nonlinear processes in turbulence which rests on three basic properties:

1. The total energy density $\int E\,dk$ is conserved by the inviscid dynamics. The characteristic time of the spectral energy redistribution for *local* interaction of turbulent scales is of order of the vortex turnover time, $1/\sqrt{k^3 E}$.

2. The steady state in forced turbulence corresponds to a constant energy cascade through the inertial range of scales which is described by the Kolmogorov spectrum,

$$E = C\,P^{2/3}\,k^{-5/3}, \qquad (2)$$

where P is the energy flux (constant in t and k) and $C \approx 1.6$ is the Kolmogorov constant.

3. When the wave-number range is truncated to a finite value and both forcing and dissipation are absent, turbulence reaches a thermodynamic equilibrium state characterized by equipartition of energy over the wavenumber space [2]. In terms of the one-dimensional energy spectrum this means $E \propto k^2$ which is obviously a steady state solution of the equation (1) for $\nu = 0$.

Even simpler first order differential equation was proposed by Kovazhny, which can model the cascade properties but ignores the thermodynamic distributions. Due to their simplicity, Leith and Kovazhny models and their generalisations to include other physical effects turned out to be very efficient. In particular, they were used for description of mixed cascade-thermo solutions corresponding to the bottleneck effect [3], superfluid turbulence in ^{3}He with friction against the normal component [4], influence of heavy particles on turbulent suspensions [5].

Differential models have also been proposed and used before for wave turbulence. In particular, Hasselmann and Hasselmann [6] and, independently, Iroshnikov [7] (thereafter HHI) proposed a model for the spectrum of deep-water surface gravity waves, which is a fourth-order nonlinear differential equation. The order of this equation is higher than in the Leith model because, in addition to the direct energy cascade and the thermodynamic energy equipartition, it describes the inverse cascade and the equipartition state of waveaction, – an additional invariant of the water wave system. Recently this model was used to describe a sandpile-like behaviour energy cascades of waves in discrete k-space [8]. Similar model was proposed for turbulence of Kelvin waves on thin quantum vortices, and it was used to study sound radiation by such waves [9]. This model is also of the fourth order because the Kelvin wave system also conserves waveaction [1]).

II. Differential Model for 2D turbulence. A 2nd-order differential model for 2D turbulence was proposed by Lilly [19], see equation (11) below. It has both the energy and the enstrophy cascade solutions, but it ignores the thermodynamic equilibria. As we will see later, this causes a significant defect in description because it fails to predict the important difference in values of the Kolmogorov constants for the direct and the inverse cascade spectra. The minimal model that would include both cascades and the thermodynamic solutions has to be of the fourth order. In some sense, such model should be a hybrid of the Leith model, since it has to respect the Navier-Stokes scalings, and the HHI model, because of presence of the additional invariant – enstrophy. We therefore postulate the following 4th-order evolution equation for the energy spectrum in 2D turbulence,

$$\frac{\partial E}{\partial t} = a\sqrt{x}\frac{\partial^2}{\partial x^2}\Big[\sqrt{E^5 x^{19/2-D}}\frac{\partial^2}{\partial x^2}\Big(\frac{\sqrt{x^{D-1}}}{E}\Big)\Big] - \nu x E, \tag{3}$$

where $x = k^2$ and D is the dimension of the system. Although our main object is 2D turbulence, $D = 2$, we would like to keep D as a parameter for the purposes which will be clear later. For $\nu = 0$ this equation conserves the mean energy density,

$$\frac{1}{2}\langle v^2\rangle = \int E\,dk = \int \frac{E\,dx}{2\sqrt{x}}, \tag{4a}$$

and the mean enstrophy density,

$$\frac{1}{2}\langle(\nabla\times v)^2\rangle = \int k^2 E\,dk = \frac{1}{2}\int \sqrt{x}\,E\,dx. \tag{4b}$$

Equation (3) can be written as a continuity equations for the energy, E, and the enstrophy, $\Pi = k^2 E$, spectra:

$$\frac{\partial E}{\partial t} + \frac{\partial P}{\partial k} = 0\,, \quad \frac{\partial P}{\partial t} + \frac{\partial Q}{\partial k} = 0\,, \tag{5}$$

where P and Q are the flux of energy and the flux of enstrophy respectively,

$$P = -\frac{a}{2}\frac{\partial R}{\partial x}, \quad Q = -\frac{a}{2}\Big(R - x\frac{\partial R}{\partial x}\Big)\,, \tag{6a}$$

$$R = \sqrt{E^5\, x^{19/2-D}}\,\frac{\partial^2}{\partial x^2}\Big(\frac{\sqrt{x^{D-1}}}{E}\Big). \tag{6b}$$

III. Steady state solutions. Equation (3) has a general thermodynamic Raleigh-Jeans solution in which both the energy flux and the enstropy flux are zero. It is determined by the condition $R = 0$ which yields,

$$E = (T\,k^{D-1})/(k^2+\mu)\,, \tag{7}$$

where constants T and μ are temperature and chemical potential respectively.

In addition, Eq. (3) has two cascade solutions: an inverse energy cascade solution ($P = \text{const}$),

$$E = C_P\,(-P)^{2/3}\,k^{-5/3}, \tag{8a}$$

and a forward enstrophy cascade state ($Q = \text{const}$),

$$E = C_Q\,Q^{2/3}\,k^{-3}, \tag{8b}$$

where C_P and C_Q are Kolmogorov constants.

Substituting Eqs. (8) into Eqs.(6), one can find relations between C_P, C_Q and a:

$$72 = a\,C_P\,(3D+2)\Big(3D-4\Big)\,, \tag{9a}$$

$$8 = a\,C_Q\,D(D+2). \tag{9b}$$

This implies the following relation between the Kolmogorov constants,

$$\frac{C_Q}{C_P} = \Big[\frac{(3D+2)(3D-4)}{9\,D(D+2)}\Big]^{2/3}. \tag{9c}$$

Note that this expression turns into zero for $D = 4/3$ which means that for this dimension the inverse cascade rate turns into zero and the corresponding inverse-cascade spectrum degenerates into thermodynamic energy equipartition. Correspondingly, one can expect for $D = 4/3$ the Gaussian statistics for this range of scales (up-scale from the forcing scale).

For $D = 2$ we have

$$\frac{C_Q}{C_P} = \Big(\frac{2}{9}\Big)^{2/3} \approx 0.367. \tag{10}$$

Note that $D = 2$ is rather close to the critical dimension $D = 4/3$ and, therefore, statistics of turbulence in the inverse-cascade range should be expected to be quite close to Gaussian. This observation was previously made in [10]. Another consequence of the fact that $D = 2$ is close to the critical dimension $D = 4/3$ is that C_Q is significantly less than C_P as seen in (10). Experimental values for C_Q are 1.4 ± 0.3 [11] whereas DNS give 1.6 ± 0.1 [13] and more recently 1.9 ± 0.1 [14]. For C_P, experiments give $C_P \approx 6.5 \pm 1$ [12] which is consistent with DNS (e.g. $C_P \approx 7$ in [16]). More recently it was argued however [15] that the energy condensation at largest scales makes it difficult to measure C_P and its fit varies from 4.5 near the forcing scale to about 7.2 near the peak of the spectrum. Taking a ratio corresponding to most recent high-resolution DNS we have $C_Q/C_P \approx 1.9/6 \approx 0.32$ which is in a good qualitative agreement with (10). Note that although there is still a

[1]) Waveaction is an adiabatic invariant (approximately) conserved by weakly nonlinear systems whose resonant interactions are of even order. For example, on the deep water the leading resonances are four-wave whereas for the Kelvin waves they are six-wave.

significant spread in the available experimenta and numerical data on the Kolmogorov constants, their ratio is always significantly less than one and our model predicts and explains this behavior.

IV. Two-flux solutions. Pure single cascade states (8) are mathematical idealisations corresponding to infinitely wide inertial ranges, whereas in both numerical simulations and nature these ranges are finite and typically both the energy and the enstrophy cascades are present in important turbulent scales. In particular, such two-cascade states are often applied for explaining the Nastrom-Gage spectrum observed in atmospheric turbulence: with -3 exponent at low wavenumbers and $-5/3$ at the high wavenumber end [17–20]. To find a analytical expression for such two-flux spectra, Lilly introduced a second-order equation model [19] which ignores the thermodynamic states but describes both the inverse and the forward cascade states,

$$\frac{\partial E}{\partial t} = b\,\frac{\partial}{\partial k}\left[\sqrt{k\,E}\,\frac{\partial}{\partial k}\left(k^3 E\right)\right] - \nu k^2 E, \tag{11}$$

where b is a dimensionless constant which has to be found from an experimental or numerical value of C_P (using C_Q is less desirable due to certain lack of precision in finding this constant related to nonlocality and respective log-correction of the direct cascade state). We have,

$$b = 3/(4\,C_P^{3/2}) \approx 0.05. \tag{12}$$

Equation (11) is written in a form of the energy continuity equation. It can also be re-written as an enstrophy continuity equation,

$$\frac{\partial\,(k^2 E)}{\partial t} = b\,\frac{\partial}{\partial k}\left[\sqrt{k^{23/3}\,E}\,\frac{\partial}{\partial k}\left(k^{5/3}E\right)\right] - \nu k^4 E. \tag{13}$$

Such a 2nd-order model is less realistic than the 4th-order model because it gives $C_P = C_Q$ due to absence of thermodynamic states. As we will see below, this also makes it less realistic for description of the two-flux states.

The general steady-state solution of (11) is

$$E = k^{-3}\left(C_P^{3/2}|P|k^2 + C_Q^{3/2}Q\right)^{2/3}. \tag{14}$$

This spectrum exhibits behaviour characteristic of the Nastrom-Gage spectrum: -3 exponent at the left and $-5/3$ right sides of the inertial range [17, 18]. Formally, $C_P = C_Q$, and substituting this equality into (14) we would recover a solution originally obtained by Lilly [19] (whose main goal was to explain the Nastrom-Gage spectrum). However, we emphasize that the more precise 4th-order model, as well as the DNS and experimental data, give significantly different values for C_P and C_Q, and therefore we can expect that Eq. (14) to work better with C_P and C_Q taken from the 4th-order model, or from the available numerical or experimental data discussed above. Transition between the two exponents occurs at

$$k_* \sim \sqrt{C_Q^{3/2}Q\,\big/\,C_P^{3/2}|P|}\,. \tag{15}$$

As it was pointed out in [19], this transition occurs without any sinks of turbulence in between of the two sources.

Formula (15) gives correct asymptotics for $P \gg Q$ and for $P \ll Q$. However, by substituting it into more precise fourth-order model we see that both the energy and the enstropy fluxes (6a) experience order of magnitude variation in the region $P \sim Q$ instead of remaining approximately constant as they should. Thus, we conclude that (15) does not work well in the transitional region $P \sim Q$ and a better description should be expected from finding the two-flux states directly in the 4th-order model (3). Using Eqs. (6), we can write the following differential equation for the general two-flux steady state,

$$\sqrt{E^5\,x^{15/2}}\,\frac{\partial^2}{\partial x^2}\left(\frac{\sqrt{x}}{E}\right) = -2(Px + Q)/a. \tag{16}$$

This equation cannot be solved analytically, but it can be easily integrated numerically for any particular values of P and Q and compared to observational or DNS data.

Note, however, that the two-source scenario is not the only suggested explaination of Nastrom-Gage spectrum and, in particular, the steeper slope at small wavenumbers can appear due to non-universality because of ineficiency of large-scale dissipation resulting in condensation of energy at the large scales (see detailed discussion and references in review [21]).

V. Effect of friction. In many situations, particularly atmospheric turbulence and laboratory experiments, the bottom friction is believed to have an important effect on 2D turbulence, and in this section we will address this issue using our differential model. Our interest here is partially motivated by a question if the bottom friction can help to form a spectrum of the Nastrom-Gage shape, particularly in the view of results of [4] where a spectrum with transition from the -3 to the $-5/3$ exponent was obtained in 3D superfluid turbulence with friction (and see comments at[2)]). For all that follows one could use the fourth order equation, but we

[2)] Similar question was examined by Elef Gkioulekas in his talk at the Warwick Symposium workshop " Universal features in turbulence: from quantum to cosmological scales" (December 5-10, 2005). However he used a 3D differential model which is less relchose to work with the second order model here because of easier algebra. Adding friction and ignoring viscosity in (11) we have,

$$\frac{\partial E}{\partial t} = b\,\frac{\partial}{\partial k}\left[\sqrt{k\,E}\,\frac{\partial}{\partial k}\left(k^3 E\right)\right] - \gamma E, \qquad (17)$$

where γ is a k-independent friction frequency. One could try a power law substitution, $E = Ck^y$, for which the power counting immediately gives $y = -3$ [23], but the constant C turns out to be undefined (infinite). This is natural because the -3 shape coincides with the direct cascade spectrum in the absence of friction and, thus, the nonlinear term in (17) turns into zero. However, there should be a corrected solution which corresponds to a direct enstrophy cascade in presence of friction. Close to the forcing scale k_F this correction can be found substituting $E = C_Q\,Q^{2/3}\,k^{-3}(1+\delta_Q)$ into (17) which gives after linearisation,

$$\delta_Q = -\frac{\gamma}{2b} C_Q^{1/2}\,Q^{1/3}\ln(k/k_F). \qquad (18)$$

One can see that this correction is negative and growing in magnitude along the cascade to high k's. Far from the forcing scale, $k \gg k_F$ we gain a logarithmic factor,

$$E = \frac{\gamma^2 C_Q^3}{9\,k^3}\left[\ln\left(\frac{k_Q}{k}\right)\right]^2, \qquad (19)$$

where k_Q is a constant wavenumber at which an abrupt cutoff of the spectrum occurs. Note that the finite cutoff is a result of using the differential model and it indicates that in more general models the spectrum, although always non-zero, decays at these scales faster than any power of k. The value of k_Q can be obtained via numerical solution for the full spectrum matching the regions close and far from the forcing scale. Note that because neither γ nor Q contain the length dimension the expression for k_Q must contain the forcing wavenumber – the only relevant quantity containing the length dimension.

Result (19) about the log-correction is natural keeping in mind the degeneracy of coincidence of the direct cascade exponent, -3, with the exponent arising from dimensional balancing of the friction and the inertial terms in the energy balance equation. However, papers [28, 29] present a numerical and experimental evidence that the friction effect on the direct cascade is to modify the spectrum exponent rather that to log-correct it. Plausible explanations for the descrepancy with the prediction of the differential model could be: (i) energy condensation and nonlocal interactions with large scales may have been important in [28, 29] (they are not described by the differential model which is super-local) or (ii) insufficient fitting interval (one decade) in [28, 29] did not allow to distinguish between the log and power corrections. Thus, additional numerical experiments at higher resolution are desirable to obtain a larger fitting interval.

Summarising results of (18) and (19), we conclude that friction leads to a faster decay of the inverse cascade spectrum and its eventual abrupt cutoff. This means in particular that it would never take a flatter -5/3 slope at high k characteristic for Nastrom-Gage spectrum.

Let us now find the friction correction to the inverse cascade spectrum close to the forcing scale by substituting $E = C_P\,P^{2/3}\,k^{-5/3}(1+\delta_P)$ into (13) (with added friction term as in (17)). After linearisation, we have

$$\delta_P = -\frac{9\gamma}{8b}\,C_P^{1/2}\,P^{1/3}k^{-2/3}. \qquad (20)$$

One can see that this correction is negative and growing in magnitude along the cascade to low k's. Easy to see by inspection that the inverse cascade spectrum also has a finite cutoff close to which we have

$$E = \frac{\gamma^2}{4b^2}(k - k_P)^2. \qquad (21)$$

Thus we see that in the inverse cascade range one could never have slope -3 at low k characteristic for Nastrom-Gage spectrum. Exact position of the cutoff can be found only by matching the large-scale and the small-scale parts of the solution, which can be done numerically. However, up to an order-one factor, it can be uniquely found from the dimensional argument which gives,

$$k_P \sim \lambda^{3/2}/|P|^{1/2}. \qquad (22)$$

In fact, existence of such an abrupt cutoff was suggested by Lilly [22] that the Kolmogorov relation between the spectrum and the energy flux persist. This assumption is valid only when friction affects the flux only weakly, and therefore it is natural that Lilly's expression for the spectrum agrees with ours near the forcing scale (20) but it fails to predict correct behaviour near the cutoff scale (21). Estimate (22) was also obtained by Manin who considered stability of large-scale vortices [27]. This cutoff scale was also discussed in experimental works [24–26] as well as in numerical works ([21] and references therein) in the context of arrest of the inverse energy cascade by bottom friction.

Combining the results for the direct and the inverse cascade spectra, we conclude that when forcing is present at only one scale the Nastrom-Gage spectrum could not be explained by the friction effect neither in the direct cascade nor in the inverse cascade ranges. Thus, 2D turbulence with friction is fundamentally different from the 3D case, and the origin of this difference can be attributed to presence of the enstrophy cascade (with

evant for this case than the 2D model introduced in the present paper and therefore we re-examine this question using our model.

exponent coinciding with the one obtained by the formal power counting in the frictional system) and to the fact that the energy cascades in 2D and 3D are in the opposite directions with respect to each other.

Summary. In this Letter, we presented a differential models for 2D turbulence based on the 4th-order Eq. (3). Based on Eq. (3) we derived the ratio of the Kolmogorov constants 0.37 which qualitatively agrees with an experimentally and numerically measured value of about 0.32. Of particular importance is the fact that the model predicts the inverse-cascade constant to be significantly greater than the direct-cascade constant. Based on a reduced 2nd-order model (11), we discussed a mixed two-cascade solution (14) which was previously suggested by Lilly for explaining the observed Nastrom-Gage spectrum of atmospheric turbulence. We showed that, inspite of giving correct asymptotic expressions in the small and large k limits, this solution is inaccurate in the intermediate range. We concluded that the 4th-order model should be used instead for finding the two-cascade states, and doing so leads to the 2nd-order nonlinear ODE (16) which can be solved numerically in each particular case with given values of P and Q. We also examined the effect of friction on the inverse and direct cascades, and showed that in the case with only one forcing region present, the friction cannot lead to the Nastrom-Gage shape. Instead, for both cascades friction acts to reduce the spectra and to terminate them abruptly at finite cutoff wavenumbers, which agrees with previous numerical and experimental studies. However, our result that friction acts to introduce a log-correction on the direct cascade rather that to modify its exponent is different from the conclusions of previous studies [28, 29]. A possible explanation of this descrepance is that non-local effects were important in simulations and experiments of [28, 29].

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