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Higher Equations of Motion in $N = 1$ SUSY Liouville Field Theory

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Similarly to the ordinary bosonic Liouville field theory, in its $N = 1$ supersymmetric version an infinite set of operator valued relations, the "higher equations of motions", hold. Equations are in one to one correspondence with the singular representations of the super Virasoro algebra and enumerated by a couple of natural numbers (m, n). We demonstrate explicitly these equations in the classical case, where the equations of type $(1, n)$ survive and can be interpreted directly as relations for classical fields. The general form of higher equations of motion is established in the quantum case, both for the Neveu-Schwarz and Ramond series.

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1. Motivations. In ref. [1] it has been shown that in the Liouville field theory (LFT) an infinite set of relations holds for quantum operators. They are parameterized by pairs of positive integers (m, n) and called conventionally the "higher equations of motion" (HEM), because the first one $(1, 1)$ coincides with the usual Liouville equation of motion. These equations relate different basic LFT primary fields $V_a(x)$ (operators V_a can be thought of as regularized version of the exponential $\exp(2a\phi)$ of the basic Liouville field ϕ). The equations are "derived" on the basis of two conjectures. First one is the vanishing of all singular vectors in the representations built on the exponential fields. This is a natural continuation of the easily verified relations in the classical LFT, and also a mandatory requirement imposed on LFT by the unitarity. The second conjecture is much less justified and states basically that the set of exponential fields $\{V_a\}$ (with complex a allowed) covers in some sense the whole variety of primary fields in LFT. One of the open problems here is the concept of the space of local fields and its completeness. In LFT, unlike the more familiar rational conformal field theories [2], the operator state correspondence doesn't hold literally, thus making difficult a straightforward inheritance of the completeness from the unitary space of physical states. Because of this conceptual problem the status of the second conjecture in unclear and waits for a mathematically correct formulation. For the moment we simply take for granted that the space of primary local fields, the norms and completeness regardless, is spanned by an appropriate subset of $\{V_a\}$.

Higher equations turned to be useful in practical calculations. In particular, in [3] they were used to derive general four-point correlation function in the minimal Liouville gravity (see [4] for the dictionary) with one degenerate matter field. It's very likely that HEM's are potentially important in the general program of explicit construction of the complete set of correlation functions in the minimal Liouville gravity.

It is the purpose of this note to reveal a similar set of higher equations in the supersymmetric Liouville field theory (SLFT). The $N = 1$ SUSY version of LFT [5] is known to be closest partner of the bosonic theory, having very similar properties. In particular, the existence of the supersymmetric version of HEM's is naturally expected.

2. Higher equations in classical SLFT. We begin with the classical equations of motion in the $N = 1$ SLFT [6, 7]

$$\bar{\partial}\psi_{\mathrm{c}} = -iM\bar{\psi}_{\mathrm{c}}e^{\varphi}\ ;\quad \partial\bar{\psi}_{\mathrm{c}} = iM\psi_{\mathrm{c}}e^{\varphi};$$
$$\partial\bar{\partial}\varphi = iM\psi_{\mathrm{c}}\bar{\psi}_{\mathrm{c}}e^{\varphi} + M^2e^{2\varphi}\ , \tag{1}$$

where φ is the boson and $(\psi_{\mathrm{c}}, \bar{\psi}_{\mathrm{c}})$ the Majorana fermion components[1)]. As in [1] we use complex 2D coordinates $z = x + iy$ and $\bar{z} = x - iy$ ($\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$) and introduce a redundant parameter M for the sake of convenience. Equations (1) can be obtained as the extremum conditions for the following classical Lagrangian density

$$S_{\mathrm{cl}} = \frac{1}{2}\left(\partial\varphi\bar{\partial}\varphi + \psi_{\mathrm{c}}\bar{\partial}\psi_{\mathrm{c}} + \bar{\psi}_{\mathrm{c}}\partial\bar{\psi}_{\mathrm{c}} + 2iM\psi_{\mathrm{c}}\bar{\psi}_{\mathrm{c}}e^{\varphi} + \right.$$
$$\left. + M^2e^{2\varphi}\right). \tag{2}$$

[1)]Index is attached to the fields ψ_{c}, and also to the classical supercurrent S_{c} and to the stress tensor T_{c}, to distinguish them from differently normalized quantum fields, which appear in the subsequent sections.

The classical (as well as the quantum) SLFT has been introduced and studied in [6 – 8] shortly after it appeared in the string context in [5]. Here we recapitulate only those properties of the classical theory which will be of further use.

The superconformal invariance at the classical level is equivalent to the statement that the components of the classical supercurrent

$$S_{\mathrm{c}} = -i\psi_{\mathrm{c}}\partial\varphi + i\partial\psi_{\mathrm{c}}\ ;\quad \bar{S}_{\mathrm{c}} = -i\bar{\psi}_{\mathrm{c}}\bar{\partial}\varphi + i\bar{\partial}\bar{\psi}_{\mathrm{c}} \tag{3}$$

are holomorphic $\bar{\partial}S^{(\mathrm{c})} = 0$ and antiholomorphic $\partial\bar{S}^{(\mathrm{c})} = 0$ respectively. These relations, as well as similar statements $\bar{\partial}T^{(\mathrm{c})} = \partial\bar{T}^{(\mathrm{c})} = 0$ about the stress tensor components

$$\begin{aligned} T_{\mathrm{c}} &= -\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}\partial^2\varphi - \frac{1}{2}\psi_{\mathrm{c}}\partial\psi_{\mathrm{c}} \\ \bar{T}_{\mathrm{c}} &= -\frac{1}{2}(\bar{\partial}\varphi)^2 + \frac{1}{2}\bar{\partial}^2\varphi - \frac{1}{2}\psi_{\mathrm{c}}\bar{\partial}\psi_{\mathrm{c}} \end{aligned} \tag{4}$$

are easily verified to be consequences of the equations (1). To fully describe the supersymmetry we need also classical generators G and $\bar{G}$, the right and left supercharges. These operators act on the classical fields and satisfy the standard relations

$$G^2 = \partial;\ \bar{G}^2 = \partial;\ \{G, \bar{G}\} = 0. \tag{5}$$

Their action on the fundamental components φ and $(\psi_{\mathrm{c}}, \bar{\psi}_{\mathrm{c}})$ is

$$G\varphi = i\psi_{\mathrm{c}};\quad \bar{G}\varphi = i\bar{\psi}_{\mathrm{c}}. \tag{6}$$

The action of the supercharges on a general exponential field follows directly from (6) and the algebra (5)

$$Ge^{\sigma\varphi} = i\sigma\psi_{\mathrm{c}}e^{\varphi};\quad \bar{G}e^{\sigma\varphi} = i\sigma\bar{\psi}_{\mathrm{c}}e^{\varphi}. \tag{7}$$

All three equations (1) follow from the statement

$$G\bar{G}\varphi = iMe^{\varphi} \tag{8}$$

and the algebra (5). This gives, after a simple reconing

$$G\bar{G}e^{\sigma\varphi} = iM\sigma e^{(1+\sigma)\varphi} - \sigma^2\psi_{\mathrm{c}}\bar{\psi}_{\mathrm{c}}e^{\sigma\varphi}, \tag{9}$$

a relation useful in the subsequent calculations.

The classical D-operators form an infinite series $D^{(\mathrm{c})}_{2k-1}$, $k = 1, 2, \ldots$. Few first representatives read

$$\begin{aligned} D^{(\mathrm{c})}_1 &= G, \\ D^{(\mathrm{c})}_3 &= G\partial + S_{\mathrm{c}}, \\ D^{(\mathrm{c})}_5 &= G\partial^2 + 2T_{\mathrm{c}}G + 3S_{\mathrm{c}}\partial + 2\partial S_{\mathrm{c}}, \\ D^{(\mathrm{c})}_7 &= G\partial^3 + 8T_{\mathrm{c}}G\partial + 4\partial T_{\mathrm{c}} + \\ &\quad + 18T_{\mathrm{c}}S_{\mathrm{c}} + 6S_{\mathrm{c}}\partial^2 + 8\partial S_{\mathrm{c}}\partial + 3\partial^2 S_{\mathrm{c}}, \\ &\ldots \end{aligned} \tag{10}$$

There is, of course, the identical series of the "left" operators $\bar{D}^{(\mathrm{c})}_{2k-1}$. One only needs the SUSY algebra (5) the definitions of the supercurrent (3) and stress tensor (4) to verify the identities

$$D^{(\mathrm{c})}_{2k-1}e^{-(k-1)\varphi} = \bar{D}^{(\mathrm{c})}_{2k-1}e^{-(k-1)\varphi} = 0. \tag{11}$$

Direct calculation with a help of the equations of motion (1) gives

$$\begin{aligned} \bar{D}^{(\mathrm{c})}_1 D^{(\mathrm{c})}_1\varphi &= -iMe^{\varphi}, \\ \bar{D}^{(\mathrm{c})}_3 D^{(\mathrm{c})}_3\varphi e^{-\varphi} &= iM^3e^{2\varphi}, \\ \bar{D}^{(\mathrm{c})}_5 D^{(\mathrm{c})}_5\varphi e^{-2\varphi} &= -4iM^5e^{3\varphi}, \\ \bar{D}^{(\mathrm{c})}_7 D^{(\mathrm{c})}_7\varphi e^{-3\varphi} &= 36iM^7e^{4\varphi}\ . \end{aligned} \tag{12}$$

This allows to conjecture that for general $k = 1, 2, \ldots$

$$\bar{D}^{(\mathrm{c})}_{2k-1}D^{(\mathrm{c})}_{2k-1}\varphi e^{(1-k)\varphi} = i(-)^k[(k-1)!]^2M^{2k-1}e^{k\varphi}. \tag{13}$$

We will show in the subsequent sections that this is a classical limit of a (subset of) more general set of relations, the HEM's of the quantum SLFT.

3. Quantum SLFT. We remind very briefly the essence of the quantum SLFT (see [9 – 11]). The Lagrangian density is[2)]

$$\begin{aligned} \mathcal{L}_{\mathrm{SLFT}} = \frac{1}{8\pi}\left(\partial_a\phi\right)^2 + \frac{1}{2\pi}\left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right) + \\ + 2i\mu b^2\bar{\psi}\psi e^{b\phi} + 2\pi b^2\mu^2 e^{2b\phi}. \end{aligned} \tag{14}$$

Here μ is a scale coupling called conventionally the cosmological constant while b is a quantum parameter, the classical limit corresponding to $b \to 0$. Convenient combination

$$Q = b^{-1} + b \tag{15}$$

is called traditionally the background charge. In the form (14) the action is explicitly supersymmetric, the classical one (2) being achieved straightforwardly in the limit $b \to 0$ with $b\phi \to \varphi$, $b\psi \to \psi_{\mathrm{c}}$, $2\pi\mu b^2 \to M$ and

$$\int \mathcal{L}_{\mathrm{SLFT}}d^2x \to \frac{1}{2\pi b^2}S_{\mathrm{cl}}. \tag{16}$$

For the "perturbed CFT" interpretation the last two terms in (14) are better understood through the field

$$-G\bar{G}e^{b\phi} = b^2\psi\bar{\psi}e^{b\phi} - 2i\pi\mu b^2e^{2b\phi}\ , \tag{17}$$

a "top" component of an appropriate supermultiplet.

2)In this paper we always use the component form of the supersymmetric expressions, systematically avoiding superspace notations. Presently the supefield notations do not give essential economy, neither notational, nor technical.

SLFT is a superconformal field theory (SCFT), the symmetry being generated by the holomorphic and antiholomorphic components of the supercurrent

$$S(z)=-i\phi\partial\psi+iQ\partial\psi;\ \bar S(\bar z)=-i\phi\bar\partial\bar\psi+iQ\partial\bar\psi. \quad (18)$$

In the classical limit they apparently turn to the fields (3) as $S\to b^{-2}S_{\rm c}$, $\bar S\to b^{-2}\bar S_{\rm c}$. In the same way the classical holomorphic and antiholomorphic stress tensor components (4) are the limits $T\to b^{-2}T_{\rm c}$, $\bar T\to b^{-2}\bar T_{\rm c}$ of the holomorphic and antiholomorphic quantum fields

$$\begin{aligned} T(z)&=-\frac{1}{2}(\partial\phi)^2+\frac{Q}{2}\partial^2\phi-\frac{1}{2}\psi\partial\psi,\\ \bar T(\bar z)&=-\frac{1}{2}(\bar\partial\phi)^2+\frac{Q}{2}\bar\partial^2\phi-\frac{1}{2}\psi\bar\partial\psi. \end{aligned} \quad (19)$$

Together with the supercurrent (1) they form the superconformal algebra of operator product expansions

$$\begin{aligned} S(z)S(z')&=\frac{\widehat c}{(z-z')^3}+\frac{T(z')}{z-z'}+\text{reg},\\ T(z)S(z')&=\frac{3S(z')}{2(z-z')^2}+\frac{\partial S(z')}{z-z'}+\text{reg},\\ T(z)T(z')&=\frac{3\widehat c}{4(z-z')^4}+\frac{2T(z')}{(z-z')^2}+\frac{\partial T(z')}{z-z'}+\text{reg}, \end{aligned} \quad (20)$$

where the central charge is related to b as

$$\widehat c=1+2Q^2. \quad (21)$$

In terms of the Laurent components

$$T(z)=\sum_n L_n z^{-n-2}\ ;\quad S(z)=\sum_k G_k z^{-k-3/2} \quad (22)$$

(index $k\in Z+1/2$ in the Neveu-Schwarz (NS) representations and $k\in Z$ in the Ramond (R) ones) the algebra takes the conventional form of super Virasoro algebra (SV)

$$\begin{aligned} \{G_k,G_l\}&=2L_{k+l}+\frac{\widehat c}{2}\left(k^2-\frac{1}{4}\right)\delta_{k+l},\\ [L_n,G_k]&=\left(\frac{n}{2}-k\right)G_{n+k},\\ [L_m,L_n]&=(m-n)L_{m+n}+\frac{\widehat c}{8}(m^3-m)\delta_{m+n}. \end{aligned} \quad (23)$$

Local fields form the highest weight representaitons of the right and left algebras $SV\otimes\overline{SV}$, both either Neveu-Schwarz or Ramond ones. The corresponding highest weight vectors are the SCFT primary fields, denoted V_a for the NS representations and $R_a^{\pm}$ for R representations. Their dimensions depend on the parameter a as

$$\Delta_{\rm NS}(a)=\frac{a(Q-a)}{2}\ ;\quad \Delta_{\rm R}(a)=\Delta_{\rm NS}(a)+\frac{1}{16}\ , \quad (24)$$

differently for the NS and R sectors. It is not a bad idea to compare the basic SLFT fields ϕ and $(\psi,\bar\psi)$ with free massless boson and Majorana fermion. In this dictionary at $\mathrm{Re}\,a<Q/2$ the primary field V_a corresponds to the normal ordered exponential $:\exp(a\phi):$. Another convenient parameter $\lambda=Q/2-a$ is often used instead of a. Correspondingly $\Delta_{\rm NS}(a)=(\widehat c-1)/16-\lambda^2/2$ and $\Delta_{\rm R}(a)=\widehat c/16-\lambda^2/2$. In the last Ramond case there are two equally good highest weight vectors, forming a two dimensional representation $2G_0R_a^{\varepsilon}=(i-\varepsilon)\lambda R_a^{-\varepsilon}$; $2\bar G_0R_a^{\varepsilon}=(i+\varepsilon)\lambda R_a^{-\varepsilon}$ of the Cartan subalgebra $G_0^2=\bar G_0^2=L_0-\widehat c/16$ and $\{G_0,\bar G_0\}=0$. In the free field language R_a^+ and R_a^- can be related respectively to the fields $\sigma:\exp(a\phi):$ and $\mu:\exp(a\phi):$, where σ and μ are the standard order and disorder spin fields with respect to the free fermion, familiar from the Ising model [12]. In our further development this doubling of the Ramond primary fields is not very important and we will often omit the index $\pm$ near $R_a^{\pm}$, keeping however in mind this feature.

4. Degenerate primaries. At certain special values of the parameter a the $SVir$ representations are singular. This happens at [13] $a=a_{m,n}$ (or, equivalently, at $a=Q-a_{m,n}$), where $a_{m,n}=-\lambda_{m-1,n-1}$ and (m,n) is a pair of positive integers. We introduced a convenient notaiton

$$\lambda_{m,n}=(mb^{-1}+nb)/2. \quad (25)$$

In general at $a=a_{m,n}$ one singular vector appears at level $mn/2$ in the Verma module, either over $V_{a_{m,n}}=V_{m,n}$ (at $m-n\in 2Z$) or over $R_{a_{m,n}}=R_{m,n}$ ($m-n\in 2Z+1$). It is convenient to introduce for each pair (m,n) a "singular vector creation operator" $D_{m,n}$, which is graded polynomials in G_{-k} and L_{-k} of level $mn/2$ with coefficients a functions of the central charge parameter ${}^2b$, such that the singular vector appears when $D_{m,n}$ is applied to $V_{m,n}$ or $R_{m,n}$, whichever appropriate. In the NS case the normalization is unambiguously fixed through the coefficient near the highest order term $D_{m,n}=G_{-1/2}^{mn}+\ldots$. Apparently the fermion parity of this operator is that of the product mn. In the R case mn is always even, while G_0 always allows to choose $D_{m,n}$ bosonic. Let us agree to put all the fermion operators G_{-n} to the right from the bosonic ones L_{-n}, arranging each group in the order of increasing index $-n$. Then an unambiguous normalization $D_{m,n}=L_{-1}^{mn/2}+\ldots$ is prescribed by the coefficient near $L_{-1}^{mn/2}$. At moderate (m,n) the polynomial $D_{m,n}$ can be carried out manually. Here we list few first examples.

Level 1/2: A singular module is over $V_{1,1} = V_0$ with the singular vector crated by

$$D_{1,1} = G_{-1/2}. \qquad (26)$$

Level 1: A singular vector in the module over $R_{1,2}$ with

$$D_{1,2} = L_{-1} - \frac{2b^2}{1+2b^2} G_{-1} G_0 \qquad (27)$$

appears at $a = a_{1,2} = -b/2$. There is similar singular module over $R_{2,1}$. It is however needless to write down $D_{2,1}$ separately, as it is obtained from $D_{1,2}$ through the symmetry $m \leftrightarrow n$, $b \to b^{-1}$. Henceforth we will systematically omit such mirror images without any special warnings.

Level 3/2: There is an NS singular vector over $V_{1,3}$ with

$$D_{1,3} = L_{-1} G_{-1/2} + b^2 G_{-3/2}. \qquad (28)$$

Level 2: A singular representation over $R^{\pm}_{1,4}$ with the creation operator

$$D_{1,4} = L^2_{-1} + \frac{3b^2}{2} L_{-2} + \\ + \frac{b^2(1-6b^2)}{1+4b^2} G_{-2} G_0 - \frac{4b^2}{1+4b^2} L_{-1} G_{-1} G_0, \qquad (29)$$

and yet another degenerate representation to type NS, where

$$D_{2,2} = L^2_{-1} + \frac{(1+b^2)^2}{2b^2} L_{-2} - G_{-3/2} G_{-1/2}. \qquad (30)$$

Level 5/2: An NS degenerate field $V_{1,5}$ with

$$D_{1,5} = L^2_{-1} G_{-1/2} + 2b^2(1+3b^2) G_{-5/2} + \\ + 3b^2 G_{-3/2} L_{-1} + 2b^2 L_{-2} G_{-1/2}. \qquad (31)$$

Further expressions are rather cumbersome and we do not quote them here, although explicit form of $D_{m,n}$ is used below up to level 9/2. Of course the same expressions with $G_n \to \bar{G}_n$ and $L_n \to \bar{L}_n$ give the "left" creation operators $\bar{D}_{m,n}$.

In SLFT, like in the bosonic LFT, all singular vectors "decouple", i.e., in the sense of quantum operators

$$D_{m,n}(V,R)_{m,n} = \bar{D}_{m,n}(V,R)_{m,n} = 0, \qquad (32)$$

where for the primary field stands either NS or Ramond one, dependent in an obvious way on the parity of m and n. This is the quantum version of the classical relations (11). Precisely as in the case of LFT, equations (32) can be considered as the basic dynamic principle of SLFT. Even the very first steps in the study of SLFT citeArvis, DHoker, Babelon, as well as the later achievements in constuction of a consistent theory [6 – 8] are based on this "decoupling".

In the next section we state certain algebraic property of the $D_{m,n}$ operators, a supersymmetric generalization of the product formula of [1] for the "norms of logarithmic primaries" (defined more precisely below).

5. Norms of logarithmic primaries. Related to every $D_{m,n}$ of the previous section, define the "conjugate" operator $D^{\dagger}_{m,n}$ through the prescriptions $G^{\dagger}_n = G_{-n}$ and $L^{\dagger}_n = L_{-n}$. Then $D^{\dagger}_{m,n} D_{m,n}$ obviously acts invariantly at the levels. In particular the highest weight vector $|a\rangle$ (here we use unified notation for the NS and Ramond primary states) of dimension Δ ($\Delta = \Delta_{\mathrm{NS}}(a)$ or $\Delta = \Delta_{\mathrm{R}}(a)$, dependent on the type of the reperesentaion) is an eigenvector $D^{\dagger}_{m,n} D_{m,n} |a\rangle = d_{m,n}(a,b) |\Delta\rangle$ with certain eigenvalue $d_{m,n}(a,b)$. By definition this function is zero at $a = a_{m,n}$ (or $a = Q - a_{m,n}$) where Δ_a becomes the Kac dimension [13] of singular representation. We're interested in the quantity $r_{m,n}$, the coefficient in linear term

$$d_{m,n}(a,b) = r_{m,n}(a - a_{m,n}) + O\left((a-a_{m,n})^2\right). \qquad (33)$$

This coefficient is a function of b. "Manual" calculations with the explicit expressions (26)–(31) (and further up to level 9/2), produce the following compact results

$$\begin{aligned}
&r_{1,1} = b^{-1}(1+b)^2\,; \quad r_{1,2} = b^{-1}(1-b^4),\\
&r_{1,3} = -b^{-1}(1-b^4)(1+3b^2),\\
&r_{1,4} = -2b^{-1}(1-b^4)(1-9b^4),\\
&r_{2,2} = b^{-5}(1-b^4)^2(1+b^2),\\
&r_{1,5} = 4b^{-1}(1-b^4)(1-9b^4)(1+5b^2),\\
&r_{1,6} = 12b^{-1}(1-b^4)(1-9b^4)(1-25b^4),\\
&r_{2,3} = b^{-7}(1-b^4)^3(1-9b^4),\\
&r_{1,7} = -36b^{-1}(1-b^4)(1-9b^4)(1-25b^4)(1+7b^2),\\
&r_{1,8} = -144b^{-1}(1-b^4)(1-9b^4)(1-25b^4)(1-49b^4),\\
&r_{2,4} = -2b^{-9}(1-b^4)^3(1-9b^4)^2(1+2b^2),\\
&r_{1,9} = 579b^{-1}(1-b^4)(1-9b^4)(1-25b^4)(1-49b^4)\times\\
&\quad\times(1+9b^2),\\
&r_{3,3} = -3b^{-13}(1-b^4)^4(1+b^2)(9-b^4)(1-9b^2).
\end{aligned} \qquad (34)$$

All of them fit into the following "product formula"

$$r_{m,n} = 2^{1-mn} \prod_{(k,l)\in[m,n]} (kb + lb^{-1})\,, \qquad (35)$$

where symbol $[m,n]$ denotes either $[m,n]_{\rm NS}$, or $[m,n]_{\rm R}$, dependent on the type of representation. Here

$$\begin{aligned}[m,n]_{\rm NS} &= \{1-n:2:n-1, 1-m:2:m-1\}\cup \\ &\cup\{2-n:2:n, 2-m:2:m\}\setminus\{0,0\}, \end{aligned} \tag{36}$$

$$\begin{aligned}[m,n]_{\rm R} &= \{1-n:2:n-1, 2-m:2:m\}\cup \\ &\cup\{2-n:2:n, 1-m:2:m-1\}\setminus\{0,0\}. \end{aligned} \tag{37}$$

In these expressions $a:d:b$ (from a to b step d) stands for the "linear" set, i.e., the set of numbers $a, a+d, a+2d, \ldots, b$, $\{A,B\}$ for the set of pairs (k,l) with k and l running independently the sets A and B, $\{A_1,B_1\}\cup\{A_2,M_2\}$ for the standard union of two sets. Finally, $\ldots\setminus\{0,0\}$ means that the pair $(0,0)$ is excluded.

Expression (35) is very much like the one obtained in [1] for the similar characteristic related to the singular representations of the usual Virasoro algebra. In that case a line of "physical" arguments has been proposed, based on the consistency of HEM's with the one point functions in the so called "Poincaré disk geometry" (see [14]). At the same time it is clear that the product formula is of pure algebraic nature and has nothing to do neither with HEM's nor with the Poincaré disk. It is desirable, therefore, to have a pure algebraic derivation of the product formula, as well as of its SUSY version (35). It is plausible that such a derivation can be found studying the structure of moduli embeddings in the nontrivial case of rational b^2 (authors thank B. Feigin for a discussion of this point). Mention also an algebraic, althouth rather complicated proof of a particular case of the product formula, related to the singular representations $(1,n)$ of the Virasoro algebra [15].

In this short note we follow a simplified way, opposite to that of [1]. In the absence of a direct proof, we take eq.(35) for granted and derive the coefficients in HEM's comparing it with the one point functions on the Poincaré disk [16, 17]. This procedure, unlike the study of multipoint functions in [1], makes analysis very compact and allows to avoid heavy calculations with the SLFT stucture constants.

6. Logarithmic degenerate fields and HEM's. Now we're in the position to define, in the spirit of ref.[1], the set of "logarithmic degenerate fields" $V'_{m,n}$ $(m-n\in 2Z)$ and $R_{m,n}$ $(m-n\in 2Z+1)$. Defining general logarithmic fields $V'_a=\partial V_a/\partial a$ and $R'_a=\partial R_a/\partial a$ as the derivatives in a (a normal ordered free fields $:\phi\exp(a\phi):$ and $\sigma(\mu):\phi\exp(a\phi):$ is what they look like in the $\phi\to-\infty$ free field limit), let us set

$$\begin{aligned} V'_{m,n} &= V'_a|_{a=a_{m,n}}, \quad m-n\in 2Z, \\ R'_{m,n} &= R'_a|_{a=a_{m,n}}, \quad m-n\in 2Z+1. \end{aligned} \tag{38}$$

Whereas general $V'_{m,n}$ and $R'_{m,n}$ are logarithmic fields (as well as general V'_a and R'_a), holds true the following

Proposition:

$$\bar D_{m,n}D_{m,n}(V,R)'_{m,n} \tag{39}$$

are primary fields.

We will not repeat the proof here, as it repeats lierally the considerations of ref.[1]. Similarly to the Liouville case, in SLFT these primary fields are to be identified with other exponential fields discussed in sect.3. Comparing dimension we find

$$\bar D_{m,n}D_{m,n}(V,R)'_{m,n} = B_{m,n}(\tilde V,\tilde R)_{m,n}, \tag{40}$$

where $\tilde V_{m,n}=V_{a_{m,-n}}$ and $\tilde R_{m,n}=R_{a_{m,-n}}$ have dimensions $\Delta^{\rm (NS)}_{m,n}+mn/2$ and $\Delta^{\rm (R)}_{m,n}+mn/2$ respectively.

Equations (40) are our long anticipated SLFT HEM's. The remaining problem of the numerical coefficients $B_{m,n}$ is discussed in the next section. We will do this comparing right and left hand sides of (40) inside correlation functions. The simplest one is the one-point function in the non-compact geometry of the Poincaré disk. In this geometry, unlike sphere or "finite disk" [18], the gauge group $SL(2,R)$ is an isometry and therefore there is no problem of factorizing its orbits. There are thus all reasons to expect all operator-valued relations to hold already on the one point function level. On the other hand the such one-point functions are relatively simple [16, 17]. They will be discussed in the next section.

7. One point functions on the Poincaré disk. In refs.[16, 17] the one point functions are constructed in the so called Poincaré disk geometry. Roughly speaking, this geometry is a quantum version of the "basic" classical solution to the classical SLFT equations of motion (1) inside the unit disk $|z|<1$

$$e^{\varphi} = \frac{2}{M^2(1-z\bar z)^2}\ ; \quad \psi=\bar\psi=0\,. \tag{41}$$

The object of study is the one point funcitons of the basic exponential SLFT fields (the meaning of the index (m,n) near the one point functions, boundary states and amplitudes is explained in [14] and [16, 17])

$$\langle (V,R)_a\rangle_{(1,1)} = \frac{\langle {\rm B}_{(1,1)}|(V,R)_a\rangle}{\langle {\rm B}_{(1,1)}|V_0\rangle} = U^{\rm (NS,R)}_{(1,1)}(a)\ , \tag{42}$$

where we denoted as $\langle {\rm B}_{(1,1)}|$ the boundary state radiated by from the absolute of the Poincaré disk. In this paper we will not repeat the considerations of refs.[16, 17], quoting only the net result

$$U_{(1,1)}^{(\mathrm{NS})}(a)=\frac{\left[\pi\mu\gamma\left(\frac{Qb}{2}\right)\right]^{-a/b}\Gamma\left(\frac{Qb}{2}\right)\Gamma\left(\frac{Q}{2b}\right)Q}{2(Q-2a)\Gamma\left(\frac{Qb}{2}-ab\right)\Gamma\left(\frac{Q}{2b}-\frac{a}{b}\right)}, \tag{43}$$

$$U_{(1,1)}^{(\mathrm{R})}(a)=\frac{\left[\pi\mu\gamma\left(\frac{Qb}{2}\right)\right]^{-a/b}\Gamma\left(\frac{Qb}{2}\right)\Gamma\left(\frac{Q}{2b}\right)Q}{2\Gamma\left(\frac{1}{2}+\frac{Qb}{2}-ab\right)\Gamma\left(\frac{1}{2}+\frac{Q}{2b}-\frac{a}{b}\right)}.$$

Substituting equations (40) to these one point functions, we obtain

$$\left\langle \mathrm{B}_{(1,1)}|\bar{D}_{m,n}D_{m,n}V(R)'_{m,n}\right\rangle = B_{m,n}\left\langle \mathrm{B}_{(1,1)}|(\tilde{V},\tilde{R})_{m,n}\right\rangle. \tag{44}$$

The boundary state $\langle \mathrm{B}_{1,1}|$ enjoys superconformal invariance. This means that for all $n\in Z$ and all $k\in Z$, $Z+1/2$ the following identities hold

$$\begin{aligned}\left\langle \mathrm{B}_{(1,1)}\right|\bar{G}_k &= -i\left\langle \mathrm{B}_{(1,1)}\right|G_{-k}, \\ \left\langle \mathrm{B}_{(1,1)}\right|\bar{L}_n &= \left\langle \mathrm{B}_{(1,1)}\right|L_{-n}\,.\end{aligned} \tag{45}$$

It is easy to see that these identities entail (operator $D^\dagger_{m,n}$ is defined in sect. 5)

$$\begin{aligned}\left\langle \mathrm{B}_{(1,1)}\right|\bar{D}_{m,n} &= \left\langle \mathrm{B}_{(1,1)}\right|D^\dagger_{m,n}, \qquad mn\in 2Z, \\ \left\langle \mathrm{B}_{(1,1)}\right|\bar{D}_{m,n} &= -i\left\langle \mathrm{B}_{1,1}\right|D^\dagger_{m,n}, \qquad mn\in 2Z+1.\end{aligned} \tag{46}$$

In the last case a multiplier $-i$ remains because at $mn\in 2Z+1$, unlike all others, the singular vector creating operator is fermionic (odd in G). We find

$$\begin{aligned}&\left\langle \mathrm{B}_{(1,1)}\right| D^\dagger_{m,n}D_{m,n}(V,R)'_{m,n}\rangle = \\ &= i^{mn-[mn/2]}B_{m,n}\left\langle \mathrm{B}_{(1,1)}|(\tilde{V},\tilde{R})_{m,n}\right\rangle\,.\end{aligned} \tag{47}$$

In terms on the one point functions Eq.(33) has the following interpretation

$$\left\langle \mathrm{B}_{(1,1)}|D^\dagger_{m,n}D_{m,n}(V,R)'_{m,n}\right\rangle = r_{m,n}\left\langle \mathrm{B}_{(1,1)}|(V,R)_{m,n}\right\rangle, \tag{48}$$

where numbers $r_{m,n}$ are from the product formulas (35),(36) and (37). Therefore

$$B_{m,n}=(-i)^{mn-[mn/2]}\frac{r_{m,n}U_{(1,1)}^{(\mathrm{NS,R})}(a_{m,n})}{U_{(1,1)}^{(\mathrm{NS,R})}(a_{m,-n})}. \tag{49}$$

This equality allows to find easily the coefficients $B_{m,n}$. Two cases should be distinguished

1. NS case (m and n both either even or odd)

$$\begin{aligned}B_{m,n} &= 2^{mn}i^{mn-[mn/2]}b^{n-m+1}\left[\pi\mu\gamma(bQ/2)\right]^n\times \\ &\times b^{n-m+1}\gamma\left(\frac{m-nb^2}{2}\right)\prod\nolimits_{(k,l)\in\langle m,n\rangle_{\mathrm{NS}}}\lambda_{k,l}.\end{aligned} \tag{50}$$

2. R case (m odd and n even)

$$\begin{aligned}B_{m,n} &= 2^{mn}b^{n-m}\left[\pi\mu\gamma(bQ/2)\right]^n\times \\ &\times\gamma\left(\frac{1}{2}+\frac{m-nb^2}{2}\right)\prod\nolimits_{(k,l)\in\langle m,n\rangle_{\mathrm{R}}}\lambda_{k,l}.\end{aligned} \tag{51}$$

Symbols $\lambda_{k,l}$ were defined in (25) while the sets $\langle m,n\rangle_{\mathrm{NS}}$ and $\langle m,n\rangle_{\mathrm{R}}$ include the following pairs of integers (k,l)

$$\begin{aligned}\langle m,n\rangle_{\mathrm{NS}} &= \{1-m:2:m-1, 1-n:2:n-1\}\cup \\ &\cup\{2-m:2:m-2, 2-n:2:n-2\}\setminus\{0,0\},\end{aligned} \tag{52}$$

$$\begin{aligned}\langle m,n\rangle_{\mathrm{R}} &= \{1-m:2:m-1, 2-n:2:n-2\}\cup \\ &\cup\{2-m:2:m-2, 1-n:2:n-1\}\setminus\{0,0\}.\end{aligned} \tag{53}$$

Only the series $(1,2k-1)$, $k=1,2,\ldots$ of HEM's allows classical limit. It is easy to check that (50) at $b\to 0$ turns to $B_{1,2k-1}\to b^{-1}(2\pi\mu b^2)^{2k-1}[(k-1)!]^2$, in agreement with the results (12) of classical calculations.

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