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The density of stationary points in a high-dimensional random energy landscape and the onset of glassy behaviour

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We calculate the density of stationary points and minima of a $N \gg 1$ dimensional Gaussian energy landscape. We use it to show that the point of zero-temperature replica symmetry breaking in the equilibrium statistical mechanics of a particle placed in such a landscape in a spherical box of size $L = R\sqrt{N}$ corresponds to the onset of exponential in N growth of the cumulative number of stationary points, but not necessarily the minima. For finite temperatures we construct a simple variational upper bound on the true free energy of the $R = \infty$ version of the problem and show that this approximation is able to recover the position of the whole de-Almeida-Thouless line.

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Statistical properties of stationary points (extrema as well as saddles) of random high-dimensional landscapes continue to attract considerable interest in the physical and mathematical literature, both analytically and numerically. Motivations come from fields of research as diverse as the study of disordered and glassy systems and string theory – see [1 – 3] and references therein. In particular, considerable effort [4] has gone into trying to understand what changes the intricate scenario of ergodicity breaking typical for mean-field spin glass models implies in the statistics of the associated free energy landscape (the so-called TAP [5] variational functional).

One of the simplest models known to display many non-trivial hallmarks of glassy dynamics and thermodynamics is the so-called toy model of a classical particle in a box of characteristic size L filled with a random potential. The position of the particle is characterized by the coordinate vector $\mathbf{x} = (x_1, ..., x_N)$, $|\mathbf{x}| \le L$, and the standard choice of the energy function is

$$\mathcal{H}(\mathbf{x}) = \frac{\mu}{2}\sum_{k=1}^{N} x_k^2 + V(x_1, ..., x_N), \qquad (1)$$

with $\mu \ge 0$. The random Gaussian-distributed potential $V(\mathbf{x})$ is assumed to have zero mean, and its covariance is chosen to have a form ensuring stationarity, isotropy, and a well-defined large N–limit:

$$\langle V(\mathbf{x}_1)\, V(\mathbf{x}_2)\rangle_V = N\, f\left(\frac{1}{2N}(\mathbf{x}_1 - \mathbf{x}_2)^2\right). \qquad (2)$$

Similarly, the natural scaling of the sample size is $L = R\sqrt{N}$ [6]. Here and henceforth the notation $\langle \ldots \rangle_V$ stands for averaging over the random potential $V(\mathbf{x})$.

In the limit when both $N \to \infty$ and $R \to \infty$ the thermodynamics of the model has enjoyed a long history of research starting from early works [7 – 9]. A detailed discussion of the limit $N \to \infty$ for the fixed box size $R < \infty$ can be found in a recent paper [6]. The paper [10] investigates the related problem for finite dimensions $N < \infty$ in a broad physical context.

The calculation in [1] revealed that the replica symmetry breaking (interpreted in the standard way as ergodicity breaking) in the zero-temperature limit of the $R = \infty$ version of the model is accompanied by the emergence of an exponentially large total number of stationary points in the energy landscape. As naively one may expect that the low-temperature thermodynamics should be dominated by minima rather than by the totality of stationary points, the issue deserves further investigation, in particular in the general $R < \infty$ case.

The goal of the present paper is to compute the density of all stationary points as well as of only minima at a given value of the potential, hence energy, in the limit $N \gg 1$ (in the simplest case $N = 1$ similar questions were addressed in [11]). Our analysis reveals that generally the domain of existence of the glassy phase with broken ergodicity (at zero temperature T) turns out to be associated with the existence of exponentially many stationary points in the energy landscape, but not necessarily exponentially many minima. In an attempt to extend our considerations to finite temperatures we also construct a simple variational functional providing an upper bound on the true free energy of the $R=\infty$ version of the problem. Surprisingly, counting stationary points in this simple-minded approximation is able to capture such a nontrivial feature as the precise position of the de-Almeida-Thouless line in the whole (μ, T) plane.

Our aim is to calculate the mean probability density $\rho_s(V, \mathbf{x})$ of the value V of the potential $V(\mathbf{x})$ taken over all stationary points of the landscape $\mathcal{H}(\mathbf{x})$ around the position $\mathbf{x}$, with the stationary points of any index counted with equal weight. We will also be able to address its counterpart, $\rho_m(V, \mathbf{x})$, where only minima of $\mathcal{H}(\mathbf{x})$ are taken into account. It is convenient to define the joint probability density $\mathcal{F}(V, \mathbf{x}, \hat{K})$ of the scalar argument V, the $N-$component vector argument $\mathbf{x}$ and the $N\times N$ real symmetric matrix argument $\hat{K}$ as

$$\mathcal{F}(V,\mathbf{x},\hat{K}) = \langle \delta\left(V - V(\mathbf{x})\right) \times \\ \times\, \delta\left(\mu\mathbf{x} + \frac{\partial}{\partial\mathbf{x}}V(\mathbf{x})\right)\delta\left(\hat{K} - \frac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}}V(\mathbf{x})\right)\Big\rangle_V, \quad (3)$$

in terms of which the density $\rho_s(V, \mathbf{x})$ can be expressed as [1]:

$$\rho_s(V,\mathbf{x}) = \int |\det\left(\mu\hat{I}_N + \hat{K}\right)| \, \mathcal{F}(V,\mathbf{x},\hat{K})\, d\hat{K}, \quad (4)$$

where $\hat{I}_N$ stands for $N\times N$ identity matrix. The density of minima $\rho_m(V, \mathbf{x})$ is given by a similar integral, but with the integrand containing the (matrix) step-function factor $\theta\left(\mu\hat{I}_N + \hat{K}\right)$ ensuring that all the eigenvalues of the matrix $\mu\hat{I}_N + \hat{K}$ are positive.

Introducing the Fourier integral representation for each of the delta-functional measures (of scalar, vector, or matrix argument) in Eq.(3) facilitates performing the ensemble average explicitly in view of the Gaussian nature of the potential Eq.(1) and its stationarity, Eq.(2). After some straightforward but long manipulations we find:

$$\mathcal{F}(V,\mathbf{x},\hat{K}) \propto \exp\left\{\frac{\mu^2\mathbf{x}^2}{2f'(0)} - \frac{V^2}{2Nf(0)}\right\} \times \\ \times \int_{-\infty}^{\infty} \exp\left\{-\frac{N}{2}t^2 - \frac{N}{4\mu_{cr}^2}\mathrm{Tr}\times \right. \\ \left. \times \left[\hat{K} - \left(gt + \frac{f'(0)}{f(0)}\frac{V}{N}\right)\hat{I}_N\right]^2\right\} dt. \quad (5)$$

Here, and henceforth, we systematically disregard various multiplicative constant factors for the sake of brevity and use the following notations:

$$\mu_{cr} = \sqrt{f''(0)}, \quad g^2 = f''(0) - \frac{f'(0)^2}{f(0)} \geq 0. \quad (6)$$

Substituting Eq.(5) back into Eq.(4), and introducing the notations

$$\hat{H} = \hat{K} - \left(gt + \frac{f'(0)}{f(0)}\frac{V}{N}\right)\hat{I}_N,$$

$$\mu_{ef}(t,V) = \mu + gt + \frac{f'(0)}{f(0)}\frac{V}{N},$$

we find that the density function factorizes as

$$\rho_s(V,\mathbf{x}) = \mathcal{N}_s\, \rho_0(\mathbf{x})\rho_s(V)\,, \quad (7)$$

where

$$\rho_0(\mathbf{x}) = \exp\left\{\frac{\mu^2\mathbf{x}^2}{2f'(0)}\right\},$$

$$\rho_s(V) = \exp\left\{-\frac{V^2}{2Nf(0)}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{N}{2}t^2\right\}\mathcal{D}_s(t,V)dt, \quad (8)$$

with

$$\mathcal{D}_s(t,V) = \int |\det\left(\mu_{ef}(t,V)\hat{I}_N + \hat{H}\right)| \times \\ \times \exp\left\{-\frac{N}{4\mu_{cr}^2}\mathrm{Tr}\hat{H}^2\right\} d\hat{H}, \quad (9)$$

and the factor $\mathcal{N}_s$ accounts for the overall normalisation. Analogously, the density of minima $\rho_m(V, \mathbf{x})$ is given by a very similar expression, the only difference being that Eq.(9) is replaced by

$$\mathcal{D}_m(t,V) = \int \exp\left\{-\frac{N}{4\mu_{cr}^2}\mathrm{Tr}\hat{H}^2\right\} \times \\ \times \det\left(\mu_{ef}(t,V)\hat{I}_N + \hat{H}\right)\theta\left(\mu_{ef}(t,V)\hat{I}_N + \hat{H}\right) d\hat{H}. \quad (10)$$

Our main interest is to extract the leading asymptotics of these expressions in the limit $N \gg 1$. To this end, for the analysis of $\rho_s(V)$ we use the results from [1], where it was demonstrated that essentially $D_s(t, V)$ can be replaced in the large N limit with

$$\mathcal{D}_s(t,V) \propto \exp N\Phi(s), \quad s = \mu_{ef}(t,V)/\mu_{cr}, \quad (11)$$

where the function $\Phi(s) = \Phi(-s)$ is given explicitly by

$$\Phi(s\geq 0) = \frac{s^2}{4} - \theta(s-2)\left[\frac{s\sqrt{s^2-4}}{4} - \ln\left(\frac{s+\sqrt{s^2-4}}{2}\right)\right]. \quad (12)$$

Perhaps, the shortest way to understand the above asymptotics is to notice that actually,

$$\Phi[s]=\int_{-2}^{2}\ln|s+\lambda|\nu_{sc}(\lambda)d\lambda,\quad \nu_{sc}(\lambda)=\frac{1}{2\pi}\sqrt{4-\lambda^2}, \tag{13}$$

where the integral is understood in the sense of the principal value. The above formula can be verified by applying ideas from statistical mechanics to the evaluation of $D_s(t,V)$. A rigorous exposition of the method and further references can be found in [12], and a related informal treatment is given in a recent insightful paper [13].

To ensure a well-defined large N limit we rescale $V\to NV$, and evaluating the integral over t with the Laplace method we find that $\rho_s(V)\propto\exp\left[-N\mathcal{S}(V)\right]$, with

$$\mathcal{S}(V)=\frac{V^2}{2f(0)}+\min_t\left[\frac{t^2}{2}-\Phi\left(\mu_{ef}(t,V)/\mu_{cr}\right)\right]. \tag{14}$$

Correspondingly, in the large N limit the density function $\rho_s(V)$ displays a pronounced maximum around the most probable value $V=V_*=\operatorname{argmin}\mathcal{S}(V)$, which can be found from the system of equations:

$$V_*=\frac{f'(0)}{\mu_{cr}}\frac{d\Phi(s)}{ds}|_{s=s_*},\quad t_*=\frac{g}{\mu_{cr}}\frac{d\Phi(s)}{ds}|_{s=s_*}, \tag{15}$$

where we have used the notation

$$s_*=[\mu+gt_*+\frac{f'(0)}{f(0)}V_*]/\mu_{cr}\equiv[\mu+\frac{\mu_{cr}^2}{f'(0)}V_*]/\mu_{cr}, \tag{16}$$

the second equality following from the obvious relation $t_*=gV_*/f'(0)$ implied by Eq.(15), and the definition Eq.(6). Moreover, for $s_*\ge 0$ Eq.(13) implies that

$$\frac{d\Phi(s)}{ds}|_{s=s_*}=\frac{1}{2}\left[s_*-\theta(s_*-2)\sqrt{s_*^2-4}\right]. \tag{17}$$

Assuming first $0\le s_*\le 2$, the first of the equations (15) together with Eqs.(16), (17) immediately yields $V_*=\mu f'(0)/\mu_{cr}^2$ and $s_*=2\mu/\mu_{cr}$, the solution being compatible with our assumption for $0\le\mu\le\mu_{cr}$. On the other hand, assuming $s_*>2$ the first of the equations (15) solved together with Eq.(17) yields the relation

$$s_*=\frac{\mu_{cr}}{f'(0)}V_*+\left(\frac{\mu_{cr}}{f'(0)}V_*\right)^{-1}, \tag{18}$$

which is consistent with our assumption. Substituting for s_* from Eq.(16) one immediately finds $V_*=f'(0)/\mu$.

Summarizing, we see that the most probable value V_* of the potential $V(\mathbf{x})$ at a stationary point of the energy surface $\mathcal{H}(\mathbf{x})$ is given by

$$V_*=\frac{f'(0)}{\mu}\,\theta(\mu-\mu_{cr})+\mu\frac{f'(0)}{\mu_{cr}^2}\,\theta(\mu_{cr}-\mu). \tag{19}$$

The corresponding values of $\mathcal{S}(V_*)$ can be found from Eq.(14):

$$\mathcal{S}(V_*)=\begin{cases}-\frac{1}{2}+\ln\left(\frac{\mu_{cr}}{\mu}\right), & \mu\ge\mu_{cr}\\ -\frac{\mu^2}{2\mu_{cr}^2}\quad, & \mu<\mu_{cr}\end{cases}. \tag{20}$$

The mean total number $\langle\#_s\rangle_V$ of stationary points is obviously obtained from the density $\rho_s(V,\mathbf{x})$ in Eq.(7) by integrating it across the range of coordinates inside the sample $|\mathbf{x}|\le L=R\sqrt{N}$ and over all possible values of the potential V. The integral over V is dominated by the most probable value V_* found above, whereas the coordinate integration is easily performed by the Laplace method yielding

$$\int_{|\mathbf{x}|\le R\sqrt{N}}\rho_0(\mathbf{x})\,d\mathbf{x}\propto$$

$$\propto\exp\frac{N}{2}\begin{Bmatrix}\ln\left(2\pi R_\mu^2\right), & R>R_\mu=\frac{\sqrt{|f'(0)|}}{\mu}\\ \ln\left(2\pi R^2\right)+1-\frac{R^2}{R_\mu^2}, & R<R_\mu\end{Bmatrix}. \tag{21}$$

Collecting all exponential factors from Eqs.(20), (21) and also restoring the overall normalisation from the requirement $\lim_{\mu\to\infty}\langle\#_s\rangle_V=1$, one finds that the associated cumulative complexity of stationary points $\Sigma_s(\mu,R)=\lim_{N\to\infty}\frac{1}{N}\ln\langle\#_s\rangle_V$ is given by

$$\Sigma_s=\frac{1}{2}\theta(R_\mu-R)\left[\ln\left(\frac{R^2}{R_\mu^2}\right)+1-\frac{R^2}{R_\mu^2}\right]\le 0 \tag{22}$$

as long as $\mu\ge\mu_{cr}$, whereas for $\mu<\mu_{cr}$ the complexity $\Sigma_s(\mu,R)$ is given by

$$\Sigma_s=\frac{1}{2}\left[-1+\frac{\mu^2}{\mu_{cr}^2}-\ln\left(\frac{\mu^2}{\mu_{cr}^2}\right)\right]>0\quad\text{for}\quad R\ge R_\mu,$$

$$\Sigma_s=\frac{1}{2}\left[\ln\left(\frac{R^2}{R_{cr}^2}\right)+\frac{\mu^2}{\mu_{cr}^2}\left(1-\frac{R^2}{R_{cr}^2}\right)\right],\quad R<R_\mu, \tag{23}$$

where introduced an important lengthscale, cf.[6]:

$$R_{cr}=\sqrt{|f'(0)|/f''(0)}. \tag{24}$$

The calculation of the density of minima $\rho_m(V)$ proceeds in a similar way. The required large N asymptotics of $\mathcal{D}_m(t,V)$ in Eq.(10) can be extracted from the results of a recent paper [13], and is given by

$$\mathcal{D}_m(t,V) \propto \exp\left(-\frac{N^2}{2}\theta(2-s)\mathcal{G}(s) + N\phi(s)\right), \quad (25)$$

where s has the same meaning as in Eq.(11), $\phi(s)$ is given by the right-hand side of Eq.(12) for $s \geq 2$, and

$$\mathcal{G}(s) = \frac{1}{216}\left[72s^2 - s^4 - (s^3+30s)\sqrt{12+s^2}\right] - \\ - \log\left(\frac{s+\sqrt{s^2+12}}{6}\right). \quad (26)$$

When calculating $\rho_m(V)$ we again rescale $V \to NV$, subdivide the integration range into two pieces $s \in \in (-\infty, 2]$ and $s \in [2,\infty)$ and evaluate two resulting integrals by the Laplace method in the large N limit. The first integral $I_1(V)$ is dominated by the point where $\mathcal{G}(s)$ is minimized, i.e. by the vicinity of $s=2$, the endpoint of the integration range, where actually $\mathcal{G}=0$. Remembering that $\phi(2)=1$, we find after some manipulations $I_1(V) \propto \exp -N\mathcal{S}_1(V)$, where

$$\mathcal{S}_1(V) = \frac{1}{2g^2 f(0)}\left[\mu_{cr}V - f'(0)\left(2-\frac{\mu}{\mu_{cr}}\right)\right]^2 + \\ + \frac{1}{2}\left(2-\frac{\mu}{\mu_{cr}}\right)^2 - 1. \quad (27)$$

Evidently, $I_1(V)$ is strongly peaked about the value V_1, where

$$V_1 = \frac{f'(0)}{\mu_{cr}}\left(2-\frac{\mu}{\mu_{cr}}\right), \quad \mathcal{S}_1(V_1) = \frac{1}{2}\left(2-\frac{\mu}{\mu_{cr}}\right)^2 - 1. \quad (28)$$

In the second integral, $I_2(V)$, the arguments leading to Eqs.(19), (20) show that for any μ it is dominated by the same value $V = V_* = f'(0)/\mu$ as in the $\mu > \mu_{cr}$ case in (19), and is given by $I_2(V) \propto \exp -N\mathcal{S}(V_*)$ with $\mathcal{S}(V_*) = -1/2 + \ln(\mu_{cr}/\mu)$ (see (20)). It is easy to verify that $\mathcal{S}_1(V_1) > \mathcal{S}(V_*)$ for $\mu > \mu_{cr}$, and for $\mu < \mu_{cr}$ the inequality is reversed, giving the leading large N asymptotics as

$$\rho_m(V) \propto \begin{cases} \exp -N\mathcal{S}(V_*), & \mu > \mu_{cr} \\ \exp -N\mathcal{S}_1(V_1), & \mu < \mu_{cr} \end{cases}. \quad (29)$$

We thus find that the associated complexity of the minima, given by $\Sigma_m(\mu,R) = \lim_{N\to\infty}\frac{1}{N}\ln\langle\#_m\rangle_V$, coincides with $\Sigma_s(\mu,R)$ in the range $\mu \geq \mu_{cr}$. For $\mu < \mu_{cr}$ the counterpart of (23) is however given by a different expression:

$$\Sigma_m = \frac{1}{2}\left[1 - \left(2-\frac{\mu}{\mu_{cr}}\right)^2 - \ln\left(\frac{\mu^2}{\mu_{cr}^2}\right)\right] > 0 \quad (30)$$

as long as $R \geq R_\mu$, whereas for $R < R_\mu$ it is given by

$$\Sigma_m = \frac{1}{2}\left[2 - \frac{R^2}{R_{cr}^2}\frac{\mu^2}{\mu_{cr}^2} + \ln\left(\frac{R^2}{R_{cr}^2}\right) - \left(2-\frac{\mu}{\mu_{cr}}\right)^2\right]. \quad (31)$$

Let us first compare (23) and (30), (31) in the infinite-sample limit $R=\infty$ studied in [1]. In that case, we see that *both* an exponentially large number of stationary points, and exponentially many minima co-exist in precisely the same interval $0 < \mu \leq \mu_{cr}$. Moreover, it is precisely the same interval of the μ-axis known to correspond at zero-temperature to the glassy phase with broken ergodicity [7, 8, 6].

Recent analysis of the general $R < \infty$ case of our model by the replica trick [6] has revealed that at zero temperature the phase with broken ergodicity covers the strip $R_{cr} < R < \infty$, $0 < \mu < \mu_{cr}$ in the (R,μ) plane. As is easy to infer from the last line in expression Eq.(23), the above strip is exactly the domain supporting positive cumulative complexity $\Sigma_s > 0$. At the same time, analysis of (30) shows that $\Sigma_m > 0$ for any $0 < \mu < \mu_{cr}$ only when the sample radius satisfies $R > R_m = e\,R_{cr} > R_{cr}$. On the other hand, if $R_{cr} < R < R_m$ then for the values of μ satisfying $0 < \mu < \mu_- < \mu_{cr}$ with

$$\mu_- = \frac{2\mu_{cr}}{1+R^2/R_{cr}^2}\left[1 - \sqrt{1 - \frac{1+R^2/R_{cr}^2}{2}\ln\frac{R_m}{R}}\right] \quad (32)$$

the complexity of minima Σ_m is negative, although we have seen that $\Sigma_s > 0$ in this range of parameters. We thus conclude that the replica symmetry – hence, ergodicity, – breaking implies that the landscape supports exponentially many stationary points, but not necessarily minima.

It is natural to try to understand whether similar relations exist at finite temperatures for the *free* energy landscape. For the present model the full TAP [5] variational functional is as yet unknown, and constructing it we consider as an interesting problem for a future work. To this end, we propose below a simple variational functional which provides only an upper bound for the true free energy of our problem. Surprisingly enough, we find that such a functional already contains enough information to reproduce the position of the de-Almeida-Thouless line in the whole (μ,T) plane.

Concentrating, for simplicity, on the infinite sample $R = \infty$ case, we employ for our goal the Gibbs–Bogoliubov–Feynman[14] inequality:

$$F(H) \leq F(H_a) + \langle H - H_a \rangle_{H_a} \quad (33)$$

valid for any two Hamiltonians H and H_a, with $F(H) = -T \ln \mathrm{Tr}\, e^{-H/T}$ and $F(H_a) = -T \ln \mathrm{Tr}\, e^{-H_a/T}$ standing for the corresponding free energies and

$$\langle (\ldots) \rangle_{H_a} = \mathrm{Tr}\left[(\ldots) e^{-H_a/T}\right] / \mathrm{Tr} e^{-H_a/T}. \quad (34)$$

The role of H in our consideration will be played by $\mathcal{H}(\mathbf{x})$ from Eq.(1), and we choose H_a in the form $H_a(\mathbf{x}) = \frac{\mu}{2}(\mathbf{m} - \mathbf{x})^2$ with the N-component vector $\mathbf{m}$ playing the role of a variational parameter. For this choice, the inequality Eq.(33) takes the form

$$F(\mathcal{H}) \leq F_a(\mathbf{m}) = \frac{NT}{2} \ln \frac{\mu}{T} + \frac{\mu}{2} \mathbf{m}^2 + V_a(\mathbf{m}), \quad (35)$$

where

$$V_a(\mathbf{m}) = \int V\left(\mathbf{m} + \mathbf{y}\sqrt{\frac{T}{\mu}}\right) \exp\left\{-\frac{1}{2}\mathbf{y}^2\right\} d\mathbf{y}\,, \quad (36)$$

with $d\mathbf{y} = \prod_{i=1}^{N} \frac{dy_i}{\sqrt{2\pi}}$. The problem of finding the best possible approximation to the true free energy in this class of trial Hamiltonians obviously amounts to minimizing $F_a(\mathbf{m})$ with respect to the parameter $\mathbf{m}$. Note that $V_a(\mathbf{m})$ is obviously a random Gaussian function with zero mean. The corresponding covariance function can be calculated using Eq.(2) and Eq.(36) as

$$\langle V_a(\mathbf{m}_1)\, V_a(\mathbf{m}_2) \rangle_V = N\, f_a\left(\frac{1}{2N}(\mathbf{m}_1 - \mathbf{m}_2)^2\right), \quad (37)$$

where

$$f_a\left(\mathbf{b}^2\right) = \int f\left[\left(\mathbf{b} + \mathbf{y}\sqrt{\frac{T}{\mu N}}\right)^2\right] \exp\left\{-\frac{1}{2}\mathbf{y}^2\right\} d\mathbf{y}, \quad (38)$$

with the function f taken from Eq.(2). The invariance of the integrand in Eq.(38) ensures that the last expression indeed depends only on $\mathbf{b}^2$. Choosing $\mathbf{b} = (|b|, 0, \ldots, 0)$, and rescaling $\mathbf{y} \to (y_1, \sqrt{N}\tilde{\mathbf{y}})$, we can easily evaluate the integral over $\tilde{\mathbf{y}}$ in the large N limit by Laplace method. This immediately leads to a very simple relation:

$$\lim_{N\to\infty} f_a\left(\mathbf{b}^2\right) = f\left(\mathbf{b}^2 + \frac{T}{\mu}\right). \quad (39)$$

Now it is evident that the free energy landscape represented by the variational functional $F_a(\mathbf{m})$ in Eq.(35) is very similar to that of the zero-temperature energy landscape $\mathcal{H}(\mathbf{x})$ from Eq.(1). In particular, the functional $F_a(\mathbf{m})$ should possess exponentially many stationary points, and also minima, as long as

$$\mu^2 < \mu_{cr}^2(T) = f_a''(0) \equiv f''\left(\frac{T}{\mu}\right), \quad (40)$$

where we recall the definition Eq.(6). The condition Eq.(40) coincides precisely with one describing the position of the de-Almeida-Thouless line in the (μ, T) plane for the present model [7, 8, 6].

When the major part of the calculations presented in this paper were already completed, we became aware of a recent preprint by A.J. Bray and D.S. Dean [15] with a quite similar scope. The authors restricted their presentation to the limiting case $\mu = 0$ of the same model, but were actually able to calculate complexity of critical points with a given index. A characteristic lengthscale L_c appearing in their paper is simply related to our $R_m = eR_{cr}$, see Eq.(32), as $L_c = \sqrt{2\pi e}\, R_m$, the extra factor accounting for the difference between their cubic and our spherical shape of the sample in the large N limit.

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