Three-loop β function of the Wess-Zumino-Witten model

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A three-loop β function of the renormalization group, which agrees with the nonperturbative results of the conformal field theory for the derived β function at the critical point, has been calculated for the Wess-Zumino-Witten model.

In this letter we present the results of a calculation of a three-loop renormalization-group β function in a two-dimensional Wess-Zumino-Witten (WZW) model, which were obtained on the basis of perturbation theory.

We assume that the field $U(x)$ has the same values as those of the compact and

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semisimple group G

$$U(x) = \exp[i\phi(x)], \quad \phi(x) = \phi^i(x) T^i, \tag{1}$$

where T^i are the generators of the G group with the structure constants f_{ijk}. We can then write the action of the WZW model in the form[1]

$$I[U] = \frac{1}{4\lambda^2} \int d^2x \, \mathrm{tr} \, \partial_\mu U \partial^\mu U^{-1} + n\Gamma[U], \tag{2}$$

where $\Gamma(U)$ is the WZW term,[2-3] and λ^2 is the coupling constant. Because Γ is a multiple-valued quantity, the condition that the quantum theory must be consistent accounts for the fact that the coupling constant in front of the WZW term is quantized. As a result, n is an integer (or a half-integer).[3]

Theory (2) can be rewritten as a nonlinear two-dimensional σ model with a torsion[3]

$$I[\phi] = \frac{1}{2\lambda^2} \int d^2x \left\{ g_{ab}(\phi) \, \partial^\mu \phi^a \partial_\mu \phi^b + \frac{2n}{3} h_{ab}(\phi) \epsilon^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b \right\}, \tag{3}$$

for which the curvature tensor R_{abcd}, which is constructed from the metric g_{ab}, the torsion $(S \sim dh) S_{abc}$, and the curvature tensor $\hat{R}_{abcd}$, in which the torsion S is taken into account, can be written[3]

$$R_{abcd} = f_{ijm} f_{klm} V_a^i V_b^j V_c^k V_d^l, \tag{4}$$

$$S_{abc} = \eta f_{ijk} V_a^i V_b^j V_c^k,$$

$$\hat{R}_{abcd} = (1 - \eta^2) f_{ijm} f_{klm} V_a^i V_b^j V_c^k V_d^l,$$

where $V_a^i(\phi)$ is the local base (the "tetrad") which is associated with the metric g_{ab} (ϕ), and $\eta = \eta\lambda^2/2\pi$.

Using current algebra, Witten[1] showed that the WZW quantum model is conformally invariant at the critical point $(\eta^2 = 1)$ (in the case of a collimated, $\hat{R}_{abcd} = 0$, group manifold), allowing Knizhnik and Zamolodchikov[4] to effectively use the conformal-field-theory methods in this case.

Since there is no conformal invariance outside the critical point, the perturbative calculation methods, which are based on the covariant background-field method for the nonlinear two-dimensional σ models with a torsion, are the principal methods.[3]

A corollary of the renormalizability of theory (2) is the structure of the l-loop covariant counterterm:

$$\Delta I^{(l)} = \frac{(\lambda^2)^{l-1}}{2} \int d^2x \, T_{ab}^{(l)} (\delta^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\mu \phi^a \partial_\nu \phi^b, \tag{5}$$

where $T_{ab}^{(l)}$ can be written in the form

$$T^{(l)}_{ab} = \sum_{n=1}^{l} \frac{1}{(2\epsilon)^n} T^{(n,\,l)}_{ab}(R, S) \,. \tag{6}$$

The quantum perturbation-theory calculations for theory (3) require that there be a ultraviolet and infrared regularizations. The latter can easily be carried out by introducing an auxiliary mass term into the theory. The use of size regularization for ultraviolet divergences requires a prescription for handling the $\epsilon^{\mu\nu}$ symbol. In fact, perturbation theory requires that only $\epsilon^{\mu\nu}\,\epsilon^{\rho\sigma}$ be defined.

Since there is no *a priori* reason to favor one prescription for ϵ symbols over another, it would be reasonable to use the most common prescription

$$\epsilon^{\mu\nu}\,\epsilon^{\rho\sigma} = C(d)\,[\,\delta^{\mu\sigma}\,\delta^{\nu\rho} - \delta^{\mu\rho}\,\delta^{\nu\sigma}\,]\,, \tag{7}$$

where the function $C(d)$ allows an expansion in a series in ϵ ($d \equiv 2 - 2\epsilon$)

$$C(d) = 1 + C_1(2\epsilon) + C_2\,(2\epsilon)^2 + \ldots, \tag{8}$$

with generally arbitrary constants C_1, C_2... . The symbol δ in (7) is, by definition, a d-dimensional symbol, in contrast with $\bar{\delta}^{\mu\nu}$, which is a two-dimensional symbol. For example, the 't Hooft-Veltman regularization,[5] in which the ϵ symbols are treated as essentially two-dimensional entities with

$$\epsilon^{\mu\nu}\,\epsilon^{\rho\sigma} = [\,\bar{\delta}^{\mu\sigma}\,\bar{\delta}^{\nu\rho} - \bar{\delta}^{\mu\rho}\,\bar{\delta}^{\nu\sigma}\,]\,, \tag{9}$$

corresponds to

$$C(d) = 4/d^2 = 1/(1-\epsilon)^2\,, \tag{10}$$

$$C_1 = 1, \quad C_2 = 3/4\,.$$

The β function of the σ model (3), which is defined as

$$\beta_{ab} = \mu \frac{d}{d\mu}\left[\frac{1}{\lambda^2(\mu)}\, g_{ab}\right]\,, \tag{11}$$

where μ is the scaling parameter of the renormalization group, is related to the counterterms by the relation[6]

$$\beta_{ab} = T^{(1,1)}_{ab} = 2\lambda^2\, T^{(1,2)}_{ab} + 3\lambda^4\, T^{(1,3)}_{ab}\,. \tag{12}$$

For the WZW model (2)–(4) the β_{ab} function can be written in the form[6]

$$\beta_{ab} = -\frac{\delta_{ab}}{\lambda^4(\mu)}\,\mu\frac{d}{d\mu}\,\lambda^2(\mu) \equiv -\frac{\delta_{ab}}{\lambda^4}\,\beta_\lambda\,, \tag{13}$$

where

$$\beta_\lambda = \mu\frac{d}{d\mu}\,\lambda^2(\mu)\,; \tag{14}$$

The WZW term cannot be renormalized in accordance with its topological nature.

The two-loop β function of the WZW model was calculated on the basis of perturbation theory by Ketov[6]:

$$\beta_\lambda = -\frac{\lambda^4 Q}{2\pi}(1-\eta^2) - \frac{\lambda^6 Q^2}{2(2\pi)^2}(1-\eta^2)(1-\eta^2-2C_1\eta^2), \tag{15}$$

where Q is the eigenvalue of the quadratic Casimir operator in the associated representation

$$f_{ijk}f_{njk} = Q\delta_{in}. \tag{16}$$

Result (15) is consistent with the fact that the theory is finite at the critical point with $\eta^2 = 1$ (Ref. 1). In (15) β_λ should be compared with the nonperturbative result[4] for the derivative of β_λ at the critical point

$$\left.\frac{\partial\beta_\lambda}{\partial\lambda^2}\right|_{\eta^2=1} = \frac{2Q}{n}\left[\frac{1}{1+Q/n}\right] = \frac{2Q}{n} - \frac{2Q^2}{n^2} + \frac{2Q^3}{n^3} + O\left(\frac{Q^4}{n^4}\right), \tag{17}$$

which agrees, within two loops, with (15) if $C_1 = 1$ is used (the t'Hooft-Veltman regularization).[6,7]

As a result of an analysis of the ultraviolet divergences of the three-loop diagrams on the basis of the background-field method for nonlinear σ models with a torsion,[3,6] the following three-loop contribution to the β function in (15) was obtained using a direct calculation (which was difficult to carry out) of the corresponding counterterms, with allowance for the necessary subtractions of the one- and two-loop subdivergences:

$$\beta_\lambda^{(3\text{-}loop)} = -\frac{Q^3\lambda^8}{(4\pi)^3}(1-\eta^2)[q_0 + (1-\eta^2)q_2 + (1-\eta^4)q_4], \tag{18}$$

where the numerical values of the coefficients q_0, q_2, and q_4 are

$$q_0 = 22 - 4C_1 - 8C_2 - 4C_1^2, \tag{19}$$

$$q_2 = \frac{1}{2^3\times 3^4}[-15545 + 4038\,C_1 + 2212\,C_2],$$

$$q_4 = \frac{1}{2^3\times 3^4}[-1053 - 1446\,C_1 - 1564\,C_2 + 2592\,C_1^2].$$

Equations (18) and (19) are the basic results. The calculations themselves will be published later.

A three-loop contribution (18) is also zero at the critical point. Remarkably, $q_0 = 8$ for the 't Hooft-Veltman regularization, exactly equal, within three loops, to the nonperturbative result (17)! The numerical values of q_2 and q_4 in this case are

$$q_2 = -\frac{1231}{81}, \qquad q_4 = -\frac{135}{81} = -\frac{5}{3}. \tag{20}$$

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