# Spin and statistics of soliton in a superfluid $^3$He-$A$ film

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A particle-like soliton with an integer topological charge $Q$ in a $^3$He-$A$ film has a spin $s = (\hbar/2)n|Q|$, where the integer parameter $n$ depends on the film thickness. In certain intervals of the film thickness, a soliton with $|Q| = 1$ thus obeys Fermi statistics.

In 2D systems with an order parameter in the form of a unit vector $\mathbf{d}(x, y, t)$, the hydrodynamic action may contain a Chern-Simons topological term or $\theta$-term $S_\theta = \hbar\theta H$, where $H$ is an integer Hopf invariant for the field $\mathbf{d}$ (Ref. 1). The Hopf index is expressed in terms of $\mathbf{d}$ through an auxiliary gauge field $A_\mu$ ($\mu = 0,1,2$):

$$H = \frac{1}{32\pi^2} \int dx dy dt e^{\mu\nu\lambda} A_\mu F_{\nu\lambda} , \tag{1}$$

$$F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu = \mathbf{d} \cdot [\partial_\nu \mathbf{d},\ \partial_\lambda \mathbf{d}] . \tag{2}$$

The $\theta$-term is responsible for the quantum statistics of particle-like solitons with a topological charge

$$Q = \frac{1}{4\pi} \int dx dy F_{12} . \tag{3}$$

When two identical solitons with a charge $Q$ are interchanged, the index $H$ changes by $Q^2$, so the partition function is multiplied by $\exp(i\theta Q^2)$. Because of unitarity, $\theta$ can take on only values[2] $\pi n$, so the elementary solitons are fermions if $n$ is odd and bosons if $n$ is even. Dzyaloshinskii *et al.*[2] suggested $n = 1$ for a spin-1/2 Heisenberg antiferromagnet, but it turned out that in such a substance there is no $\theta$-term (Ref. 3, for example). The reason is that the Hopf invariant and thus the $\theta$-term are not invariant under time and space inversion.[4]

Volovik[4] has shown that because of the combination of spin antiferromagnetism with orbital ferromagnetism in a $^3$He-film, the symmetry allows the existence of a $\theta$-term, in the form

$$S_\theta = \hbar\theta(\mathbf{l} \cdot \hat{\mathbf{z}})H , \tag{4}$$

where $\mathbf{l}$ is the orbital ferromagnetism vector in $^3$He-$A$, which is fixed along the normal $\hat{\mathbf{z}}$ to the film in the case of a film: $\mathbf{l} = \pm \mathbf{z}$. Under time reversal and also under the transformation $(x,y,z) \to (x, -y, -z)$, which plays the role of inversion in the 2D space $(x,y)$, the Hopf invariant $H$ changes sign; however, this change is offset by the rotation of the vector $\mathbf{l} \to -\mathbf{l}$, however, so the action remains invariant under these transformations. Unfortunately, the calculation of $\theta$ carried out in Ref. 4 was incorrect. In the present letter we derive $\theta = \pi n$, where $n$ is the number of transverse-quantization energy levels in the film and can be either even or odd, depending on the film thickness. Consequently, in certain intervals of the film thickness an elementary soliton (with $|Q| = 1$) will be a fermion.

This result has been derived in two ways. In the first case, we use a purely gauge SU(2) field

$$\mathbf{A}_\mu = -\mathbf{d}A_\mu + [\mathbf{d}, \partial_\mu \mathbf{d}], \tag{5}$$

This field corresponds to a 3D rotation which depends on the coordinates and the time and which sends the field $\mathbf{d}(x, y,t)$ into the uniform field $\mathbf{d}(\infty)$. A direct expansion of the partition function of a $^3$He-$A$ film in $A_\mu$ at a constant field $\mathbf{d} = \mathbf{d}(\infty)$ leads to expression (4) with $\theta = \pi n$, where $H$ is expressed in $\mathbf{A}_\mu$ in the following way:

$$H = -\frac{1}{96\pi^2} \int dxdydt\, e^{\mu\nu\lambda} \mathbf{A}_\mu \cdot [\mathbf{A}_\nu, \mathbf{A}_\lambda]. \tag{6}$$

The second method, which we will discuss here, makes use of the relationship between the effect of interest here and the analog of the quantum Hall effect which was analyzed in Ref. 5. Specifically, the Hall current in a $^3$He-$A$ film and the anomalous spin current which arises from the $\theta$-term are related to each other. One can thus express the parameter $\theta$ in terms of the Hall conductivity $\sigma_{xy}$: $\theta/\pi = 4\pi\hbar\sigma_{xy} = n$.

The anomalous spin current is found by varying the $\theta$-term in the action with respect to $\mathbf{A}_i$. We are interested in the flux of the spin projection onto the $\mathbf{d}$ axis:

$$\mathbf{d}\cdot\mathbf{j}^i_{spin} = \frac{\hbar\theta}{16\pi^2}\mathbf{d}\cdot[\mathbf{A}_j, \mathbf{A}_0]e^{ijk} l_k = \frac{\hbar\theta}{16\pi^2} e^{ijk} l_k \mathbf{d}\cdot[\partial_j\mathbf{d}, \partial_t\mathbf{d}], \quad i = 1, 2, 3. \tag{7}$$

We also make use of the Larmor theorem, according to which a spin rotation with an angular velocity $[\mathbf{d},\dot{\mathbf{d}}]$ is equivalent to the application of a magnetic field $\mathbf{H} = \frac{1}{\gamma}[\mathbf{d},\dot{\mathbf{d}}]$ to the spins, where $\gamma$ is the gyromagnetic ratio for the $^3$He nuclei. Accordingly, the following term, linear in $\mathbf{H}$ and in the gradient of the field $\mathbf{d}$, must be present in the expansion of the spin current:

$$\mathbf{d}\cdot\mathbf{j}^i_{spin} = \frac{\hbar\theta}{16\pi^2} e^{ijk} l_j \gamma \mathbf{H}\cdot\partial_k\mathbf{d}. \tag{8}$$

We will now find this term on the basis of the analogy with the quantum Hall effect. For this purpose we make use of the equivalent description of $^3$He-$A$ as a liquid which has two superfluid components: one consisting of Cooper pairs of fermions on a Fermi sphere with an "up" spin projection, and the other having a "down" spin. The spin quantization axis must be perpendicular to the vector $\mathbf{d}$, since the projection of the spin of the Cooper pairs onto the $\mathbf{d}$ axis is zero in the ordinary description of $^3$He-$A$.

In the presence of a gradient of the chemical potential, $\partial_k\mu$, according to the results of Ref. 5, each of the superfluid components carries its own Hall flux of particles, directed perpendicular to $\partial_k \mu$:

$$j_\downarrow^i = j_\downarrow^i = \frac{1}{2}\sigma_{xy} e^{ijk} l_j \partial_k \mu, \qquad \sigma_{xy} = n/2h, \tag{9}$$

where $n$ is the number of transverse-quantization levels in the film below the Fermi surface. If a magnetic field is acting on the spins and dictates the direction of the spin quantization axis, the chemical potentials for the different spin projections move away from each other:

$$\mu_\uparrow = \mu - \frac{\hbar}{2}\gamma H, \qquad \mu_\downarrow = \mu + \frac{\hbar}{2}\gamma H. \tag{10}$$

As a result, the following flux of the spin projection onto the magnetic field arises:

$$\frac{\hbar}{2}(j_\uparrow^i - j_\downarrow^i) = -\frac{n\hbar}{16\pi} e^{ijk} l_j \gamma \partial_k H. \tag{11}$$

In generalizing (11) to the case of an arbitrary direction of the magnetic field, for a field **d** which is spatially nonuniform, we need to allow for the circumstance that the local spin quantization axis is always perpendicular to **d**. Accordingly, a vector generalization of expression (11) is described by the replacement $H \rightarrow \mathbf{H}_\perp = \mathbf{H} - \mathbf{d}(\mathbf{d}\cdot\mathbf{H})$. As a result, we have the following expressions for the spin current and its projection onto the **d** axis:

$$\mathbf{j}_{spin}^i = -\frac{n\hbar}{16\pi} e^{ijk} l_j \gamma \partial_k \mathbf{H}_\perp, \qquad \mathbf{d}\cdot\mathbf{j}_{spin}^i = \frac{n\hbar}{16\pi} e^{ijk} l_j \gamma \mathbf{H}\cdot\partial_k \mathbf{d}. \tag{12}$$

Comparing (12) with expression (8), we find the unknown $\theta = \pi n$. Consequently, solitons with odd $n$ obey Fermi-Dirac quantum statistics if the film thickness is such that it corresponds to an odd number of transverse-quantization levels below the Fermi energy.

In terms of the Fermi statistics of solitons, a $^3$He–$A$ film should be nothing unique. A search should be made among magnetic media for 2D electron structures, in which a spin ferromagnetism or antiferromagnetism is combined with an orbital ferromagnetism or antiferromagnetism in such a way that the symmetry does not forbid the existence of a $\theta$-term. Analogously, there could also be 3D magnetic structures whose symmetery would allow a $\theta$-term, e.g., of the form $\int d^3x dt A_\mu J^\mu$ (Ref. 6), where $A_\mu$ is the electromagnetic field, and $J^\mu$ is the topological charge density of a particle-like soliton in 3D space. A $\theta$-term of this sort will lead to a fractional quantization of the electric charge of the solitons.

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