Critical exponents of one-dimensional models in lattice field theory

A. V. Zabrodin
Institute of Chemical Physics, Academy of Sciences of the USSR

(Submitted 20 February 1990; resubmitted 18 April 1990)
Pis'ma Zh. Eksp. Teor. Fiz. **51**, No. 11, 543–546 (10 June 1990)

The critical exponents in the long-wavelength asymptotic behavior of the correlation functions have been found in nonrelativistic one-dimensional models of lattice field theory.

1. Conformal field theory has proved to be an effective approach for studying the correlation properties of one-dimensional quantum-mechanical systems.[1–6] The basic idea here is that systems of this sort have a conformal symmetry in the long-wavelength limit, since a phase transition occurs in them at absolute zero. The correlation functions fall off in a power-law fashion at large distances. The exponents describing this behavior (the critical exponents) can be expressed in terms of the energies of the lowest-lying excitations of the system in a large but finite volume.[1]

This approach was developed in Refs. 2 and 4–6 for exactly solvable models. In the present letter we generalize it to continuum field theory models with a long-range effect. It turns out that in these cases the long-wavelength properties of the system are described by a Gaussian model: a very simple conformal theory with a unit integral charge. The spectrum of dimensionalities of a Gaussian model is known to depend on a single continuous parameter. The value of this parameter for specific models (and also all the critical exponents) are determined by the nature of the short-range interaction and are expressed in a universal fashion in terms of the velocity of massless excitations in the system.

For lattice models, however, no general relations of this sort between the critical exponents and thermodynamic propeties have been known. In the present letter we derive universal formulas for the critical exponents in terms of the theromodynamic properties of the system.

We consider a system of spin-zero particles on a one-dimensional lattice with a Hamiltonian

$$H = -\sum_{x=1}^{L} (\psi^+_{x+1}\psi_x + \psi^+_x\psi_{x+1} - 2\psi^+_x\psi_x)$$
$$+ g\sum_{x,y} \psi^+_x\psi_x V_{x-y}\psi^+_y\psi_y - \mu\sum_{x=1}^{L}\psi^+_x\psi_x, \qquad (1)$$

where ψ_x^+, ψ_x are lattice Bose or Fermi operators, V_{x-y} is a repulsive binary interaction potential ($V_x = V_{-x}$), $g > 0$ is a coupling constant, and μ is the chemical potential. We denote by ϵ_0 the energy density of the ground state of the system; then

 0021-3640/90/110614-05$01.00

$\rho = \partial\epsilon_0/\partial\mu$ is the equilibrium density of particles at absolute zero. We assume that the potential and the density are such that there is no gap in the excitation spectrum of the system. Under this condition, our arguments do not depend on the specific function V_x. It is rather difficult to formulate any exact conditions on V_x, although it is clear from qualitative considerations that there would be no gap for an extremely broad class of potentials. The boundary conditions in (1) are assumed to be periodic. It is convenient to think of the lattice as being rolled into a ring; this interpretation is used in §2.

2. The scale dimensionalities h of the Primar operators ϕ are known to be related to the energies E_L^ϕ of the lowest-lying excitations of the system in a box of length L (such that $\langle \text{vac}|\phi|\phi\rangle \neq 0$) in the following way:[1,2]

$$E_L^\phi - E_L^{vac} = 2\pi v L^{-1} h. \tag{2}$$

Here $|\text{vac}\rangle$ is the physical vacuum, v is the group velocity on the Fermi surface (the sound velocity), and $E_L^{\text{vac}} = \epsilon_0 L$ is the energy of the ground state of the system. The long-wavelength asymptotic expression for the simultaneous correlation function of the fields ϕ is

$$\langle vac\,|\,\phi(x)\,\phi(0)\,|\,vac\rangle \sim \cos(P_\phi x)\, x^{-2h}, \tag{3}$$

where P_ϕ is the momentum of the state $|\phi\rangle$, which is nonzero if there is a gap in the spectrum of the momentum operator.

To find the spectrum of dimensionalities in system (1), it is sufficient to find the energies of all the lowest-lying excitations at an accuracy level of L^{-1}. We begin with excitations which conserve the number of particles. First, if there is no energy gap, there is a very simple excitation with a momentum $\pm 2\pi/L$ and an energy $2\pi v/L$ (a single phonon). The dimensionality of the corresponding operator is therefore $h = 1$, which is the same as the canonical dimensionality of the density operator $\psi_x^+ \psi_x$. In addition, the spectrum $\epsilon(p)$ has a periodic branch with a period of $2\pi\rho$:$\epsilon(2m\pi\rho) = 0$. At large but finite L, the energies of these states, $\epsilon^{(m)}$, are no longer zero: $\epsilon^{(m)} \sim L^{-1}$. We also need to find the energies of these states $|\phi_m\rangle$, i.e., $\epsilon^{(m)}$. In systems which have Galilean invariance, the states $|\phi_m\rangle$ correspond to a motion of the system as a whole,[7,8] and the calculation of their energies is a trivial matter: $\epsilon^{(m)} = 4\pi^2 m^2 \rho L^{-1}$. There is no Galilean invariance on a lattice, and some other approach must be taken.

We seek states $|\phi_m\rangle$ in the form

$$|\,\phi_m\rangle = \sum_{\{x_j\}} \exp\Big(ip_m \sum_{j=1} x_j\Big)\, \Psi_m(x_1, \ldots, x_N) \prod_{k=1}^{N} \psi_{x_k}^+\, |\,0\,\rangle, \tag{4}$$

where $p_m = 2\pi m/L$, and $|0\rangle$ is the bare vacuum. In the case $m = 0$ we have $|\phi_0\rangle = |\text{vac}\rangle$, and Ψ_0 is the wave function of the ground state in the coordinate representation. For $m \neq 0$, Ψ_m should not be greatly different from Ψ_0, since p_m is small.

We now apply Hamiltonian H in (1) to state (4), and we choose Ψ_m in such a way that (4) is the eigenstate of this Hamiltonian with the lowest energy at a fixed total momentum of $2\pi m\rho$. It is easy to verify that in this case Ψ_m must be wave function of the ground state (in the N-particle sector) of the following Hamiltonian $H(p)$ with $p = p_m$:

$$H(p) = H + (1 - \cos p) \sum_{x=1}^{L} (\psi_{x+1}^{+} \psi_x + \psi_x^{+} \psi_{x+1})$$

$$- i \sin p \sum_{x=1}^{L} (\psi_{x+1}^{+} \psi_x - \psi_x^{+} \psi_{x+1}) . \qquad (5)$$

This Hamiltonian is found from H through the transformation $\psi_x^+ \to \exp(ipx)\psi_x^+$, $\psi_x \to \exp(-ipx)\psi_x$; $H(0) = H$. By virtue of gauge invariance, this transformation is equivalent to placing system (1) in a uniform magnetic field directed perpendicular to the plane of the ring into which the lattice is rolled (§1). The quantity p is obviously quantized ($p = p_m$), and the magnetic flux through the ring is $2\pi m$.

For the energy shift of the ground state of $H(p)$ in comparison with that of H we can write[1] (at the order of L^{-1})

$$\epsilon^{(m)} = {}^1/_2 L p_m^2 (\partial^2 \epsilon_0 / \partial p^2)_{p=0} = 2\pi v L^{-1} \pi \eta v^{-1} m^2 . \qquad (6)$$

In the absence of Galilean invariance, the momentum p and the flux density $j = i\langle \psi_{x+1}^+ \psi_x - \psi_x^+ \psi_{x+1} \rangle$ are generally not proportional to each other, and the susceptibility

$$\eta = \partial^2 \epsilon_0 / \partial p^2 = \partial j / \partial p \qquad (7)$$

introduced in (6) is a measure of the coupling of these quantities in system (1).

Letting the Hamiltonian H act on (4), we find that the energy of the state $|\phi_m\rangle$ is equal to $\epsilon^{(m)}$ in (6). Comparing with (2), we find the dimensionalities of the corresponding Primar operators to be $h_{0,m} = \pi \eta v^{-1} m^2$.

We now consider excitations which change the number of particles. In the case of Bose statistics, the addition of n particles to the system leads to an energy increase of $\frac{1}{2} n^2 \chi^{-1} L^{-1}$, where $\chi = \partial \rho / \partial \mu = \partial^2 \epsilon_0 / \partial \mu^2$ is the susceptibility which is the "dual" of η in (7). Comparing with (2), we find a new set of dimensionalities: $h_{n,0} = (4\pi v \chi)^{-1} n^2$.

A relationship of a completely general nature holds between the sound velocity v, on the one hand, and the susceptibilities η and χ, on the other. By virtue of relation (7) between the momentum and the flux density, the wave equation describing the propagation of long waves becomes $\chi \partial^2 u / \partial t^2 = \eta \partial^2 u / \partial x^2$ [$u(x, t)$ is the displacement of the particles of the medium; the quantity x here may be thought of as a continuous variable]. We then find the universal relation

$$v^2 = \eta \chi^{-1} . \qquad (8)$$

3. Combining the various types of excitations, and using (8), we thus find the complete spectrum of dimensionalities to be

$$h_{n,m} = n^2/R^2 + m^2R^2/4, \tag{9}$$

where

$$R^2 = 4\pi v\chi . \tag{10}$$

We have obtained the spectrum of dimensionalities of the Gaussian model.[9] For bosons, the numbers n and m are integers. It follows from simple arguments based on the symmetry properties of the wave function that the momentum $P = 2\pi\rho m$ of the lowest-lying states (with a zero energy in the limit $L\to\infty$), expressed in units of $2\pi\rho$, can be either an integer or a half-integer (depending on the parity of N): $(-1)^{2m} = (-1)^{N-1}$ (Ref. 7). For fermions and for even n, the value of m is an integer, while for odd n the value of m is a half-integer.

Using (3), we can write the asymptotic expression for the correlation functions without difficulty. The correlation function for fermion fields, for example, has the following behavior at $|y-x| \gg \rho^{-1}$:

$$\langle vac \mid \psi_x^+ \psi_y \mid vac\rangle \sim \cos(\pi\rho \mid x-y) \mid x-y\mid^{-2/R^2 - R^2/8}, \tag{11}$$

This behavior corresponds to $n = 1$ and $m = 1/2$ in (9). The choice $n = 1$ is dictated by the circumstance that the only intermediate states which contribute to the correlation function in (11) are those in which the number of particles differs from the vacuum number by 1. For fermions, m would then have to be a half-integer. It is easy to see that the minimum exponent is reached in the case $m = 1/2$. The other correlation functions are found in a corresponding way.

We wish to stress the conclusion that (10) describes the most general relationship between the critical exponents and the thermodynamic properties of the system. When we taken the continuum limit in (1), Galilean invariance is restored, and (10) becomes the same as an existing result.[2,7,8,10] In the case of exactly integrable magnetic materials, relation (10) is the same as the expression derived previously by other methods.[11] The critical exponents were expressed in terms of the susceptibility χ in the Hubbard model in Ref. 12.

I wish to thank V. E. Korepin, V. Ya. Krivnov, A. D. Mironov, and A. A. Ovchinnikov for discussions and useful comments.

[1)] The same result is of course found after a calculation of the energy shift by perturbation theory. It is sufficient to consider the first two orders in p to derive the result in the order of L^{-1}.

[1]J. L. Cardy, Nucl. Phys. B **270** (FS16), 186 (1986).
[2]N. M. Bogolyubov *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 405 (1986) [JETP Lett. **44**, 521 (1986)].
[3]H. W. Blöte *et al.*, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
[4]F. Alcaraz *et al.*, Ann. Phys. **182**, 280 (1988).
[5]H. J. De Vega and M. Karowski, Nucl. Phys. B **285**, (FS19), 619 (1987).
[6]G. Von Gehlen *et al.*, J. Phys. A **20**, 2577 (1987).
[7]A. D. Mironov and A. V. Zabrodin, Preprint, P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow, 1989; A. V. Zabrodin and A. D. Mironov, Zh. Eksp. Teor. Fiz. **97**, (1990), in press.
[8]F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).
[9]L. P. Kadanoff, Ann. Phys. **120**, 39 (1979).

[10]A. V. Zabrodin and A. A. Ovhinnikov, Zh. Eksp. Teor. Fiz. **88**, 1233 (1985) [Sov. Phys. JETP **61**, 728 (1985)].
[11]N. M. Bogoliubov *et al.*, Nucl. Phys. **275**, 687 (1986).
[12]N. M. Bogoliubov and V. E. Korepin, Int. J. Mod. Phys. B **3**, 427 (1989).

Translated by D. Parsons