Large gauge transformations and special orbits of the Virasoro group

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The classification of orbits of the Virasoro group is analyzed in terms of orbits of a gauge group. Orbits of a gauge group with a nontrivial number of windings correspond to special orbits of the Virasoro group.

Attempts have been made over the past few years to use a geometric approach in the construction and classification of conformal theories.[1-3] A correspondence has been established between certain types of conformal theories and certain types of coassociated orbits of a Virasoro group, but the overall program is still far from completion. In particular, there is the serious complication that orbits of the type[1] $\widehat{\text{diff}}S^1/SL^{(n)}(2,R)$; $\widehat{\text{diff}}S^1/T_{n,\Delta}$; $\widehat{\text{diff}}S^1/T_{n,\pm}$ cannot be quantized by the standard methods.

The problem of classifying coassociated Virasoro orbits has been solved independently by several authors by various approaches and at various times (Ref. 4; see also Refs. 1 and 5). Let us review the results which we will need here. The type of orbit is conveniently described in terms of its stationary subgroup (stabilizer). There are four different types of stabilizers of Virasoro orbits: S^1, $SL^{(n)}(2,R)$ [the index n here specifies how $SL(2,R)$ is immersed in the Virasoro group], $T_{n,\Delta}$ [a single-parameter subgroup of the Virasoro group, which is generated by a vector field with simple zeros (n is the number of zeros, and Δ is a continuous invariant which is related to a monodromy)], and $T_{n,\pm}$ [a one-parameter subgroup of the Virasoro group generated by a vector field with double zeros (n is the number of zeros, and $\pm$ is a discrete invariant associated with the orientation)].

We will show here how special Virasoro orbits—orbits of the type $\widehat{\text{diff}}S^1/SL^{(n)}(2,R)$; $\widehat{\text{diff}}S^1/T_{n,\Delta}$; $\widehat{\text{diff}}S^1/T_{n,\pm}$—arise upon a Hamiltonian reduction[6] from orbits of the coassociated action $SL(2,R)$ of the gauge group [a centrally expanded group of loops $L\widehat{SL}(2,R)$].

As is well known,[7] the orbits of the coassociated action of the gauge group LG are classified by the monodromy matrix $M \in G$. In other words, lying on the orbit corresponding to the given matrix are those elements (and only those elements) $J(x)$ of the Kac-Moody algebra of the $\widehat{LG}$ group which can be written in the form

$$J = k\partial g(x) g^{-1}(x), \qquad (1)$$

where k is the level of the Kac–Moody algebra, $\partial \equiv \partial/\partial x$, $x \in S_1$, and $g(x)$ is a function on S^1 with values in G such that

$$g(2\pi) = g(0) \quad M. \qquad (2)$$

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The matrix M is defined modulo conjugations in G, so the orbits of the coassociated action of the group are in a mutually one-to-one correspondence with the classes of conjugate elements in group G. If the group contains uncontractable loops, this classification of orbits is valid only for a "large" gauge group, which contains elements that are both homotopic and nonhomotopic with respect to unity. If we instead restrict the analysis to a "small" gauge group (linked with unity by the component LS), the monodromy matrix from the given class of conjugate elements corresponds to a series of orbits which differ from each other by a gauge transformation with a nontrivial number of windings.

For $SL(2,R)$, the classes of conjugate elements are enumerated by the matrices

$$\begin{aligned} &\text{a)}\ \cos\alpha+\sigma_2\sin\alpha, \qquad 0\le\alpha<\pi \\ &\text{b)}\ \pm(1+(\sigma_1\pm\sigma_2)\alpha), \qquad \alpha>0 \\ &\text{c)}\ \pm(\cosh\alpha+\sigma_1\sinh\alpha), \qquad \alpha>0 \end{aligned} \tag{3}$$

where

$$\sigma_1=\begin{pmatrix}0&0\\1&0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and the large gauge transformation with n windings is determined by the matrix

$$g_n(x)=\exp\{n\sigma_2x\}=\cos nx+\sigma_2\sin nx. \tag{4}$$

For a gauge action with an arbitrary matrix $g(x)$, the monodromy matrix transforms in accordance with

$$M\to g(2\pi)Mg^{-1}(0) \tag{5}$$

so we need analyze only the classes

$$\begin{aligned} &\text{a)}\ \cos\alpha+\sigma_2\sin\alpha, \qquad 0\le\alpha<\pi/2 \\ &\text{b)}\ 1+(\sigma_1\pm\sigma_2)\alpha), \qquad \alpha>0 \\ &\text{c)}\ \cosh\alpha+\sigma_1\sinh\alpha), \qquad \alpha>0 \end{aligned} \tag{3'}$$

instead of the classes in (3). In addition, we can include matrices of the type in (4) with half-integer n in the set of large gauge transformations.

Corresponding to monodromy matrices of the type in (3′) with zero windings are orbits which pass through an element of the algebra $L\widehat{SL}(2,R)$ which is constant on S^1. These constant elements, which correspondingly have the form

$$\text{a)}\ \frac{k\alpha}{2\pi}\sigma_2, \qquad 0 \le \alpha < \pi$$

$$\text{b)}\ \frac{k\alpha}{2\pi}(\sigma_1 \pm \sigma_2), \qquad \alpha > 0 \qquad (6)$$

$$\text{c)}\ \frac{k\alpha}{2\pi}\sigma_1, \qquad \alpha > 0\ ,$$

are chosen as representations. The representations for the orbits with $n = m/2$ windings are

$$\text{a)}\ (\frac{k\alpha}{2\pi} + \frac{km}{2})\sigma_2$$

$$\text{b)}\ (\pm\frac{k\alpha}{2\pi} + \frac{km}{2})\sigma_2 + \frac{k\alpha}{2\pi}\sigma_1 \cos mx + \frac{k\alpha}{2\pi}\sigma_3 \sin mx \qquad (7)$$

$$\text{c)}\ \frac{km}{2}\sigma_2 + \frac{k\alpha}{2\pi}\sigma_1 \cos mx + \frac{k\alpha}{2\pi}\sigma_3 \sin mx.$$

Before we carry out the reduction to Virasoro orbits, we would like to point out that stationary subgroups of the $L\widehat{SL}(2,R)$ group are generated for representations (7) by the following elements of the $L\widehat{SL}(2,R)$ algebra:

$$\text{a)}\ \sigma_2, \qquad \alpha \neq 0$$

$$\text{a}')\ \sigma_2,\ \ \sigma_1 \cos mx + \sigma_2 \sin mx,\ \ \sigma_3 \cos mx - \sigma_1 \sin mx, \qquad \alpha = 0 \qquad (8)$$

$$\text{b)}\ \pm\sigma_2 + \sigma_1 \cos mx + \sigma_3 \sin mx$$

$$\text{c)}\ \sigma_1 \cos mx + \sigma_3 \sin mx.$$

In order to determine the Virasoro orbits which correspond to gauge orbits with representations (7) in the case of a Hamiltonian reduction,[6] we need to put the representations in the following form by means of a nontrivial gauge transformation from the component of $L\widehat{SL}(2,R)$ associated with unity:

$$\begin{pmatrix} 0 & 1 \\ u(x) & 0 \end{pmatrix}. \qquad (9)$$

Corresponding to a coassociated action of the $L\widehat{SL}(2,R)$ gauge group which conserves the form of (9) is a coassociated action of the Virasoro group for the lower-triangle element. This element can thus be thought of as a representation on a Virasoro orbit. The upper-triangle element of the matrix of the $L\widehat{SL}(2,R)$ algebra, which determines an infinitesimal gauge transformation that conserves the form of (9), plays the role of a vector field line S^1, which determines the infinitesimal coassociated action of the Virasoro group on $u(x)$.

As we have already mentioned, the types of Virasoro orbits can be described in a natural way in terms of corresponding stabilizers, so the logic of our procedure is as follows: We first find a gauge transformation which puts the representations in (7) in the form in (9). We then apply this transformation to the elements of stabilizers (8). The upper-triangle component of these elements, being a stabilizer for the coassociated Virasoro action, then determines the Virasoro orbit.

For gauge orbits of type a) with $\alpha\neq 0$, a Virasoro stabilizer is generated by a single-parameter subalgebra of the Virasoro algebra that consists of vector fields which are constant on S^1, so the corresponding Virasoro orbit is of the type $\widetilde{\text{diff}}S^1/S^1$. It is easily shown that the normal form of the representation is a constant negative two-differential.

For gauge orbits of type a) with $\alpha=2$, $m\neq 0$, a Virasoro stabilizer is generated by a three-parameter subalgebra of the Virasoro algebra which consists of vector fields of the type 1 sin mx, cos mx. The corresponding Virasoro orbit is therefore of the type $\widetilde{\text{diff}}S^1/SL^{(m)}(2,R)$.

For gauge orbits of type b), the Virasoro stabilizer is generated by the vector field

$$\frac{1\pm\cos mx}{\frac{km}{2}+\frac{k\alpha}{2\pi}(1\pm\cos mx)},\quad m>0. \tag{10}$$

In the classification of Ref. 1, a vector field of this sort is of the type $J_{n,\pm}$. Orbits of type b) therefore reduce to orbits $\widetilde{\text{diff}}S^1/T_{n,\pm}$.

For gauge orbits of type c), the Virasoro stabilizers are generated by the vector field

$$\frac{\sinh\mu+\cosh\mu\cos mx}{\frac{km}{2}+\frac{k\alpha}{2\pi}\sinh\mu+\frac{k\alpha}{2\pi}\cosh\mu\cos mx}, \tag{11}$$

where $m>0$, and the parameter is chosen in such a way that we have $(k\alpha/2\pi)\sinh\mu+km/2>(k\alpha/2\pi)\cosh\mu$. According to the classification of Ref. 1, the corresponding orbit is of the type $\widetilde{\text{diff}}S^1/T_{n\Delta}$, $\Delta=k\alpha$.

We will conclude by pointing out some directions for further research. First, one might carry out a corresponding analysis of the symplectic sheets W_n of algebras which are analogs of Virasoro orbits for a Hamiltonian reduction with coassociated orbits of[6] $L\widehat{SL}(N,R)$. Second, it would be interesting to see how the classification of Virasoro orbits arises upon a reduction from a three-dimensional theory with a Chern–Simons Lagrangian on a manifold selected in the appropriate way. Vortices would correspond to a nontrivial number of windings in this case. Finally, the most interesting direction would be to quantize special Virasoro orbits in the approach of a quantum Hamiltonian reduction.

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