Mass shift of a classical charge and shift of the cyclotron frequency near conducting walls

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The effect of a conducting wall on the self-energy of a charge executing a finite (or infinite) motion in a magnetic (or electric) field is analyzed. The dependence of the shift on the integral of the unperturbed motion in a magnetic field ($v_\perp$) is not limited by the requirement $v_\perp \ll 1$. This dependence can be used to determine the relationship between the mass shift and the shift of the cyclotron frequency induced by the wall. Expressions derived here contain "retarded" and "unretarded" asymptotic expressions for the shift as particular limiting cases.

Self-effects in the classical electrodynamics of a point particle are observed both in the equations of motion (the radiation reaction force) and in the final corrections to the action of a particle. Increments arise because of a finite change caused by an external field in the (infinite) self-energy of an electron being accelerated by this field.[1,2] The self-effects depend on the surroundings of the particle: If there are conducting surfaces in the vicinity, they cause additional increments in the action, which can in a sense be interpreted as a change in mass.[3] The increased interest in effects of this type stems from recent high-precision experiments with a single electron in Penning traps (see the review by Brown and Gabrielse[4]). In such experiments, the boundaries have an important effect on the estimate of the accuracy with which the $(g-2)$-factor of the electron is measured. It was shown in Refs. 5–7 that the effect of boundaries on the spin precession frequency ω_s can be ignored, so the shift of the cyclotron frequency ω_c induced by the boundary is the only boundary-induced correction in $(\omega_s - \omega_c)/\omega_c = (g-2)/2$. It has also been found that the basic equations describing the effect are classical equations.[5,7–9]

We would like to suggest a fairly simple method for calculating corrections to the self-energy of an electron. These corrections also contain information about the shift of the cyclotron frequency. This method dates back to Refs. 1–3 and is based on a calculation of the action increment ΔW which stems from the self-effect of the charge:

$$\Delta W = \frac{e^2}{2} \int\int d\tau d\tau' \dot{x}_\alpha(\tau) \dot{x}_\beta(\tau') D^c_{\alpha\beta}(x, x')\Big|_0^{F,B}. \quad (1)$$

Here $\dot{x}_\alpha(\tau)$ is the classical 4-velocity corresponding to the unperturbed motion (unperturbed by the boundary) of the particle along the world line $x_\alpha(\tau)$, $D^c_{\alpha\beta}(x,x')$ is the causal Green's function of a photon, which incorporates the boundaries, and the symbol $|_0^{F,B}$ means that the change in the self-effect caused by the boundaries and the external field is to be taken into account.[1)]

In the case of a static, uniform field and for motion parallel to the conducting plane, the mass shift can be expressed in terms of the function

$$f(\tau - \tau') = (x(\tau) - x(\tau'))^2. \tag{2}$$

Since motion parallel to the plane means that one of the coordinates is fixed, the integral corresponding to (2) for the image charge [whose world line is $\tilde{x}_\alpha(\tau)$] is

$$(x(\tau) - \tilde{x}(\tau'))^2 = f(\tau - \tau') + R^2, \tag{3}$$

where R is twice the distance from the charge to the plane. Going through calculations as in Ref. 3, we find the following expression for the real[2)] part of self-effect (1):

$$\mathrm{Re}\Delta W = -\frac{\alpha\tau}{2}\,\frac{f''(\Delta\tau_+)}{|f'(\Delta\tau_+)|} \equiv -\Delta m\tau. \tag{4}$$

Here $\Delta\tau_+$ is the positive root of the equation

$$f(\Delta\tau) + R^2 = 0 \tag{5}$$

and is the interval of proper time of the charge between the emission of the photon and its absorption after reflection from the mirror. The shift is clearly Lorentz-invariant: In the case of inertial motions parallel to the plane, the intervals in (2) and (3), their derivatives with respect to the proper time, and also the root $\Delta\tau_+$ remain unchanged.

Motion in a magnetic field directed perpendicular to the plane. For the function f in a magnetic field we have the expression[2]

$$f(\Delta\tau) = -\gamma_\perp^2\Delta\tau^2 + 4R_c^2\sin^2\left(\frac{\kappa\Delta\tau}{2m}\right), \tag{6}$$

where $\gamma_\perp = (1 - v_\perp^2)^{-1/2}$, $R_c = p_\perp/\kappa = mv_\perp\gamma_\perp/\kappa$ is the radius of the cyclotron orbit, $\kappa = e\eta$, and η is the strength of the magnetic field. Using (4), we then find the following expression for Δm:

$$\Delta m = -\frac{\alpha\kappa}{2m}\,\frac{1 - v_\perp^2\cos\theta}{\theta - v_\perp^2\sin\theta}. \tag{7}$$

According to (5), θ is found from the equation

$$\theta^2 = 4v_\perp^2\sin^2\frac{\theta}{2} + \left(\frac{\kappa R}{m\gamma_\perp}\right)^2. \tag{8}$$

In addition to the magnetic field and the distance to the plane, the shift thus depends on the Lorentz-invariant parameter [$u_\alpha \equiv \dot{x}_\alpha(\tau)$] $v_\perp$:

$$v_\perp^2 = \frac{2(F_{\mu\nu}u_\nu)^2}{F_{\mu\nu}^2 + 2(F_{\mu\nu}u_\nu)^2}. \tag{9}$$

This parameter is the velocity of the particles which is transverse with respect to the field and which is conserved in the motion, in the frame of reference in which there is only a magnetic field. Transforming to that frame, we find the following expressions for the correction to the Lagrangian of the particle and to its energy, respectively:

$$\Delta L = -\frac{\Delta m}{\gamma_\perp}, \qquad \Delta E = v_\perp \frac{\partial \Delta L}{\partial v_\perp} - \Delta L. \tag{10}$$

Let us consider the nonrelativistic limit, under the assumption $v_\perp^2 \ll 1$, $\omega_c R \gtrsim 1$. For the increment ΔE we find, on the basis of (7), (8), and (10),

$$\Delta E = -\frac{\alpha}{2R} - \frac{\alpha}{R}\left(\cos\omega_c R - \frac{\sin\omega_c R}{\omega_c R} + \frac{1-\cos\omega_c R}{\omega_c^2 R^2}\right)\frac{v_\perp^2}{2}. \tag{11}$$

The quantity

$$\delta m = -\frac{\alpha}{R}\left(\cos\omega_c R - \frac{\sin\omega_c R}{\omega_c R} + \frac{1-\cos\omega_c R}{\omega_c^2 R^2}\right) \tag{12}$$

may thus be thought of as a correction to the mass of the particle. Terms on the order of $v_\perp^4$ have been discarded from the right side of (11). The shift of the cyclotron frequency which was derived in Ref. 5 can be explained completely on the basis of mass shift (12): $\delta\omega_c = -\omega_c \delta m/m$. Discarding terms $\sim R^{-2}$, R^{-3}, we find

$$\frac{\delta\omega_c}{\omega_c} = \frac{\alpha}{Rm}\cos\omega_c R. \tag{13}$$

This result agrees with expression (2.7) of Ref. 5, when we note that in our notation R corresponds to $2R$ in Ref. 5 and that the magnetic field is directed perpendicular to the plane. We also note that in the limit $\kappa \to 0$ we have $\theta \simeq \kappa R/m \to 0$, and this angle does not depend on $v_\perp$. We can thus take the limit $\omega_c R \to 0$ in (11) and find the correction[3)] to the energy of a charge moving along a plane at a constant velocity $v_\perp$:

$$\Delta E = \frac{-\alpha}{2R} + \left(\frac{-\alpha}{2R}\right)\frac{v_\perp^2}{2}. \tag{14}$$

[The asymptotic expression for the wave zone, $\omega_c R \gg 1$, was used in Ref. 5; the terms on the order of R^{-2} and R^{-3} which are present in (12), and which are required for deriving (14) were ignored.] Expression (14) obviously also gives us the correction to the energy in the limit $R \to 0$, $v_\perp \ll 1$, so turning off the field is equivalent to moving toward the plane. A charge in uniform acceleration has a corresponding property.[3] Of interest in this connection is the ultrarelativistic limit $R_c \gg R$, $v_\perp \sim 1$, $\gamma_\perp \to \infty$, in which we have

$$\Delta m = -\frac{3\alpha}{2R}\left(\frac{\kappa R \gamma_\perp}{2\sqrt{3}m}\right)^{1/2}. \tag{15}$$

Motion in an electric field. We consider a charge moving parallel to a plane in an electric field ϵ. In contrast with Ref. 3, we assume that the particle has a constant 4-velocity $u_\perp$, which is perpendicular to the field. In this case the shift depends on the Lorentz-invariant parameter $[\tilde{F}_{\mu\nu} = (i/2)\,\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}]$ $v_{\perp 0}$, given by

$$v_{\perp 0}^2 = \frac{u_\perp^2}{1+u_\perp^2} = \frac{2(\tilde{F}_{\mu\nu}u_\nu)^2}{\tilde{F}_{\mu\nu}^2 + 2(\tilde{F}_{\mu\nu}u_\nu)^2}. \tag{16}$$

This is the velocity of the particle in the frame of reference of the field, directed perpendicular to this field, and taken at the time at which the second component of the velocity vanishes ($v_\perp^2(\tau) = u_\perp \gamma^{-1}$, $\gamma^2 = 1 + u_\perp^2 + u_\parallel^2(\tau) \neq \text{const}$). We have ($\nu = e\epsilon$, $\gamma_0^2 = 1 + u_\perp^2$)

$$f(\Delta\tau) = u_\perp^2 \Delta\tau^2 - \frac{2m^2\gamma_0^2}{\nu^2}\left(\cosh\frac{\nu\Delta\tau}{m} - 1\right), \tag{17}$$

so that, according to (4) and (5), we have

$$\Delta m = -\frac{\alpha\nu}{2m}\,\frac{\cosh\theta - v_{\perp 0}^2}{\sinh\theta - v_{\perp 0}^2\theta}, \tag{18}$$

$$4\sinh^2\frac{\theta}{2} = v_{\perp 0}^2\theta^2 + \left(\frac{\nu R}{m\gamma_0}\right)^2, \tag{19}$$

and we should use the positive root of Eq. (19). The vanishing of $v_{\perp 0}$ does not mean the nonrelativistic limit, since the motion becomes a uniform-acceleration motion, i.e., clearly relativistic, in this case. This case was studied in detail in Ref. 3. The limit $R \to \infty$ ($v_{\perp 0} = 0$) gives us a nonvanishing shift $\Delta m = -\alpha v/2m$, in contrast with the case of a finite motion.[1] An increase in γ_0 in (19) does not lead to a "turn-off" of the field or an "approach" to the plane, since the characteristic value of θ in this case is on the order of $(\nu R/m\gamma_0)^{1/2}$. We thus find, in complete analogy with (15), the ultrarelativistic asymptotic expression for (18):

$$\Delta m \simeq -\frac{3\alpha}{2R}\left(\frac{\nu R\gamma_0}{2\sqrt{3}m}\right)^{1/2}. \tag{20}$$

We note in conclusion that the method proposed here can also be used to determine the correction to the mass and shift of the cyclotron frequency in the case in which the magnetic field is parallel to the plane. In particular, an additional factor of 1/2 appears on the right side of (13), in complete agreement with the results of Refs. 5 and 7. These and other questions will be discussed in a detailed paper.

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[1)] We are using a system of units with $c = 1$, $\hbar = 1$, and $\alpha = e^2/4\pi\hbar c$. We are using a metric in which we have $x_\alpha = (\vec{x}, ix_0)$, etc.

[2)] We are concerned with only the real part of ΔW in this paper, so we omit the "Re" below. The physical meaning of the imaginary part of ΔW is explained in Refs. 2 and 10 (see also Ref. 3).

[3)] Expression (14) can also be derived directly from (4), (5), and (10) by noting that at $\kappa = 0$ we have $f(x) = -x^2$.

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