# Acoustic turbulence

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(Submitted 16 May 1991)
Pis'ma Zh. Eksp. Teor. Fiz. **53**, No. 11, 558–560 (10 June 1991)

Acoustic turbulence has been analyzed in the case where there are two acoustic branches and the key process is the decay of a phonon of one branch into two phonons of another branch in the given case or the emission of a phonon by another phonon through Čerenkov radiation. In addition to the well-known (Kolmogorov) turbulence spectrum and the Rayleigh–Jeans distribution, the kinetic equations were found to have two more power solutions with $\epsilon_k \sim k^{-1}$ and $\epsilon_k \sim k^3$.

The acoustic turbulence in an ordinary compressible liquid was first studied by Zakharov and Sagdeev.[1] Assuming that the amplitudes of sound with different wave vectors are statistically independent, and assuming the validity of the Kolmogorov hypothesis, which states that the energy flux along the wavelengths is constant, Zakharov and Sagdeev[1] obtained, on the basis of a dimensional analysis, a power function $\epsilon_k \sim k^{-3/2}$ for the energy density $\epsilon_k$. They showed that the same result is obtained from the kinetic equation for phonons. The applicability of the kinetic equation and the results obtained by Zakharov and Sagdeev were criticized by Kadomstev and Petviashvili.[2] They showed that linear dispersion of sound, which leads to a decay along the straight line, does not give rise to randomization of phases, but rather to the formation of shock waves, which accounts for the entirely different shape of the spectrum. The key role of higher-order processes was noted in the earlier studies of Landau and Khalatnikov.[3]

This problem can be avoided if the system has two acoustic branches, and if the allowed processes are those processes which convert the vibrations of one branch into those of the other. An example of such a system is an isotropic solid in which the spectra of the longitudinal and transverse phonons are nondecay spectra

$$\omega_{1,t} = S_{1,t}\vec{q}(1-\gamma_{1,t}q^2);\ \gamma_{1,t} > 0,$$

where $\omega_{1,t}$ is the frequency, $S_{1,t}$ is the velocity, and $\vec{q}$ is the wave vector. There can occur cubic anharmonicities of the type $(\mathrm{div}\vec{u})^3$ and $(\mathrm{curl}\vec{u})^2\,\mathrm{div}\vec{u}$, where $\vec{u}$ is the displacement. The first anharmonicity is not effective since a decay of longitudinal sound into two longitudinal sounds is forbidden by the energy conservation law. The second anharmonicity gives rise to an allowed decay of the longitudinal sound into two transverse sounds.

The second example is the interaction of the longitudinal sound with spin waves in an isotropic antiferromagnet. The anharmonicity that can occur in this case is of the form $\mathrm{div}\vec{u}\,(\nabla l_\alpha)^2$, where $l_\alpha$ are the components of the $2D$ vector of transverse variation of the order parameter. In this case the phonon can decay into two spin waves, but the Čerenkov radiation is forbidden. The example of two-fluid hydrodynamics in

 0021-3640/91/110583-03$01.00  

superfluid helium is slightly more uncertain. The decay of the first sound into two first sounds, of the second sound into two second sounds, and of the first sound into two second sounds and the Čerenkov radiation of the second sound by the first sound are in principle allowed in this case.[1] In actual situations, however, one of these processes plays the key role. There is a regime in which the dominant process is the decay of the first sound into two second sounds,[5] while the remaining processes can be ignored.

The collinearity of the decay processes is not satisfied under conditions which were discussed above. The higher-order processes are then inconsequential and the kinetic equation for phonons of various kinds can be used.

We will show that in addition to the two known solutions corresponding to the Rayleigh-Jeans distribution and the Kolmogorov distribution, these equations have at least two more power-law solutions.

For definiteness we assume that only the decay $1 \to 22$ is allowed. We denote by $N(\vec{k})$ and $n(\vec{k})$ the filling numbers of phonons of type 1 and 2, and $\Omega(\vec{k}) = s_1 k$ and $\omega(\vec{k}) = s_2 k$ are their frequencies.

The kinetic equations can be written in the form

$$\dot{N}(\vec{k}) = \int d^3k_1 W(\vec{k};\vec{k}_1,\vec{k}_2)\delta(\Omega(k) - \Omega(k_1) - \Omega(k_2))(n_1 n_2 - N n_1 - N n_2), \tag{1}$$

$$\dot{n}(\vec{k}) = \int d^3k_1 W(\vec{k};\vec{k},\vec{k}_2)\delta(\Omega(k) - \Omega(k_2) - \Omega(k_1))(N_1 n + N_1 n_2 - n n_2), \tag{2}$$

where $\vec{k}_2 = \vec{k} - \vec{k}_1$, and the probability for the decay $W$ can be written in the form

$$W(\vec{k};\vec{k}_1,\vec{k}_2) = k k_1 k_2 f(k;k_1,k_2), \tag{3}$$

where $f(k;k_1,k_2)$ is a homogeneous function of degree 0 with respect to its arguments. We seek steady-state solutions of the kinetic equations of the power-law type.

$$N(\vec{k}) = Ak^s; \quad n(\vec{k}) = Bk^s. \tag{4}$$

The condition under which Eqs. (1) and (2) are soluble will then take the form

$$ad = bc, \tag{5}$$

where

$$a = \int d^3k_1 W(k;k_1,k_2)\delta(\Omega - \omega_1 - \omega_2)k_1^s k_2^s, \tag{6}$$

$$b = \int d^3k_1 W(k;k_1,k_2)\delta(\Omega - \omega_1 - \omega_2)k^s(k_1^s + k_2^s), \tag{7}$$

$$c = \int d^3k_1 W(k_1;k,k_2)\delta(\omega + \omega_2 - \Omega_1)k^s k_2^s), \tag{8}$$

$$d = \int d^3k_1 W(k_1;k,k_2)\delta(\omega + \omega_2 - \Omega_1)k_1^s(k^s + k_2^s), \tag{9}$$

Applying the Kac–Kontorovich transformation[6] $k_1 \to k^2/k^1$, $k_2 \to k_2 k/k_1$, to integrals (8), (9), we obtain

$$c = \int d^3k_1 W(k; k_1, k_2)\delta(\Omega - \omega_1 - \omega_2) k^{+8+2s} k_1^{-8-2s} k_1^s k_2^s, \tag{10}$$

$$d = \int d^3k_1 W(k; k_1, k_2)\delta(\Omega - \omega_1 - \omega_2) k^{+8+2s} k_1^{-8-2s} (k_1^s + k_2^s). \tag{11}$$

Comparing (10) and (11) with (6) and (7), we find four roots of Eq. (5), $s = -9/2, -4, -1, 0$. For $s = -9/2$ we find $s = s_1/2s_2 a$, $b = s_1/2s_2 d$. For $s = -4$ we have $a = c$, $b = d$; for $s = -1$ we find $s_1 a = s_2 b$ and $s_1 c = s_2 d$. Finally, for $s = 0$, $2d = b$ and $2c = d$. Although the integrals obtained in this case sometimes diverge, the collision integrals (1) and (2) converge when appropriate constants $a$ and $b$ are chosen, because the integrands are the same or nearly the same. The new solutions apparently involve an additional integral of the kinetic equations, given by

$$\int (n(k) - 2n(\vec{k}))d^3k.$$

Such analysis is also applicable to the case where the only process is the Čerenkov radiation. If, however, the two processes occur simultaneously, only the Kolmogorov solution, $s = -9/2$, and the Rayleigh–Jeans distribution, $s = -1$, will remain. The non-Kolmogorov solutions in the anisotropic magnetized plasma were previously found by Balk and Nazarenko.[7]

The energy density is $\epsilon_k \sim n_k k^3 \sim k^{3+s}$. In the solution $s = -9/2$ the energy is concentrated near the small values of $k$, but in the solution $s = -4$ the energy diverges logarithmically near the small and large values of $k$. The energy flux is nonlocal in scale. At $s = 0, -1$, the energy is concentrated in the short-wave region. The energy flux is local in the wave space (scale) only for the solution with $s = -9/2$. Interestingly, when $s = 0$, the energy flux is from small scales to large scales.

I wish to thank S. V. Nazarenko for a useful discussion.

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Translated by S. J. Amoretty