New restrictions on spatial topology of the universe from microwave background temperature fluctuations

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If the Universe has the topology of a 3-torus (T^3), then it follows from recent data on the large-scale temperature fluctuations, $\Delta T/T$, of the cosmic microwave background that either the minimal size of the torus is at least on the order of the present cosmological horizon or the large-scale $\Delta T/T$ pattern should have a symmetry plane (in the case of the effective T^1 topology), or a symmetry axis (in the case of the effective T^2 topology). The latter possibility can probably be ruled out by the data.

The possibility that a space-like hypersurface $t=\text{const}$ (or $\varepsilon=\text{const}$) of the Friedmann-Robertson-Walker (FRW) cosmological model may possess a nontrivial (i.e., not R^3) topology was known long ago in connection with the S^3 topology in the case of a closed Universe ($\Omega_{\text{tot}}\equiv\varepsilon_{\text{tot}}/\varepsilon_{\text{crit}}>1$). In this case, it follows from the age of the Universe and direct dynamical estimates of matter density that the corresponding topological size—the radius of the Universe—cannot be significantly smaller than the present cosmological horizon. A way to obtain small topological lengths is to assume a spatially flat 3-space with the 3-torus topology (T^3), i.e., to make the following identification of the points on the 3D flat hypersurface $t=\text{const}$:

$$x\equiv x+L_1,\quad y\equiv y+L_2,\quad z\equiv z+L_3,\quad L_1\geqslant L_2\geqslant L_3. \tag{1}$$

This possibility was considered in Ref. 1–3 with the conclusion that aL_1 should exceed 500–1000 Mpc, where $a\equiv a(t_0)$ is the present scale factor of the Universe (in this paper all scales are given for $H=50$ km/s/Mpc). If $aL_1\gg R_{\text{hor}}$, where R_{hor} is the present size of the cosmological horizon [$R\approx 12\,000$ Mpc if $a(t)\propto t^{2/3}$ now], then the effective spatial topology of the Universe becomes T^2; for $aL_2\gg R_{\text{hor}}$, it is effectively T^1. Note also that the remaining possibility to have a nontrivial topology with small characteristic sizes in an open FRW Universe[4] can virtually be ruled out, because it requires that $\Omega_{\text{tot}}\ll 1$, which contradicts observational data ($\Omega_m=0.2$–0.4 from dynamical estimates based on optical galaxies; use of IRAS galaxies yields a larger value for Ω_m).

Classical general relativity does not put any restrictions on L_i; they are just a part of initial conditions for the Universe at the singular hypersurface $a=t=0$. Different versions of the cosmological scenario with a de Sitter (inflationary) phase in the early Universe[5] do not preclude this topology. In this case, the T^3 topology may be a result of quantum creation of the Universe[6] (if this process is believed to be properly described by a WKB solution of the Wheeler–De Witt equation which damps exponentially under a potential barrier in the direction from $a=0$ to the border of a classically allowed region). However, as a result of inflation, the present topological sizes aL_i are expected to be much greater than R_{hor} with an overwhelming probability. It is also possible to generate a nontrivial topology of the ε=const hypersurface during inflation,[7] but not of the T^3 type. Thus, the prediction of the inflationary scenario is that spatial topology of the Universe should be trivial at all observable scales up to R_{hor} and even larger. Regardless of the assumption regarding the behavior of the early Universe, it becomes possible now to strongly restrict the hypothesis of the spatial T^3 topology of the Universe directly from recent observational data[8,9] on large-scale fluctuations of the cosmic microwave background radiation.[1)] Here the COBE data[9] appear to be the most appropriate. If $\Delta T/T$ fluctuations are produced by adiabatic perturbations and $a(t) \propto T^{2/3}$ since decoupling until the present time (the matter-dominated regime in a spatially flat FRW Universe), then the Sachs–Wolfe formula[11] is valid for large angles $\vartheta \geqslant 2°$ ($l \leqslant 30$):

$$\frac{\Delta T}{T}(\theta,\varphi)=\frac{1}{3}\Phi(R_{rec}/a,\theta,\varphi)=-\frac{1}{10}h(R_{rec}/a,\theta,\varphi), \tag{2}$$

where $\Phi(\vec{r})$, the gravitational potential, defines a perturbed space-time metric in the longitudinal gauge:

$$ds^2=(1+2\Phi)dt^2-a^2(t)(1-2\Phi)(dx^2+dy^2+dz^2), \tag{3}$$

$h(\vec{r})$ is a metric perturbation in the ultra-synchronous gauge [where $h_\alpha^\beta \approx h(\vec{r})\delta_\alpha^\beta$ as $t \to 0$, $\alpha,\beta=1,2,3$], and R_{rec} is the present radius of the surface of last scattering, $R_{rec} \approx 0.97\ R_{hor}$. Φ and h do not depend on t in a linear approximation.

In a 3-torus Universe, the spectrum of Fourier modes is discrete:

$$\Phi(\vec{r})=\sum_{npq}\Phi(\vec{k})e^{i\vec{k}\vec{r}},\quad \vec{k}=\left(\frac{2\pi n}{L_1},\frac{2\pi p}{L_2},\frac{2\pi q}{L_3}\right),\quad n,p,q\in Z,\quad n^2+p^2+q^2\neq 0 \tag{4}$$

(the $n=p=q=0$ mode is a gauge mode). $\Phi(\vec{k})$ is not assumed to have a flat (Harrison–Zeldovich) spectrum (in contrast with Ref. 10). Expanding $e^{i\vec{k}\vec{r}}$ in terms of normalized spherical harmonics, we obtain

$$\frac{\Delta T}{T}=\sum_{lm}\left(\frac{\Delta T}{T}\right)_{lm}Y_{lm}(\theta,\varphi),\quad \left(\frac{\Delta T}{T}\right)_{lm}=\frac{4\pi}{3}\sum_{npq}i^l j_l(kR_{rec}/a)Y_{lm}^*(\vec{k}/k),$$

$$k=|\vec{k}|,\quad j_l(v)=\sqrt{\frac{\pi}{2v}}J_{l+1/2}(v). \tag{5}$$

Total multipole amplitudes $(\Delta T/T)_l$ follow from the expression

$$\left(\frac{\Delta T}{T}\right)_l^2\equiv\frac{1}{4\pi}\sum_{m=-l}^{l}\left(\frac{\Delta T}{T}\right)_{lm}^2$$

$$=\frac{2l+1}{9}\sum_{\vec{k}}\sum_{\vec{k}'}\Phi(\vec{k})\Phi(\vec{k}')j_l(kR_{rec}/a)j_l(k'R_{rec}/a)P_l\left(\frac{\vec{k}\vec{k}'}{kk'}\right). \tag{6}$$

Let $aL_1 \ll R_{\rm hor}$ (a small 3-torus Universe). Then $(\Delta T/T)_l^2 \propto (2l+1)$ (this result was already obtained in Ref. 12). In particular, for a stochastic isotropic δ-correlated spectrum of perturbations $\langle \Phi(\vec{k})\Phi(\vec{k}')\rangle = \Phi^2(k)\delta_{\vec{k}\vec{k}'}$, Eq. (6) takes the form

$$\left(\frac{\Delta T}{T}\right)_l^2 = \frac{2l+1}{9}\sum_{\vec{k}} \Phi^2(k) j_l^2(kR_{\rm rec}/a) \approx \frac{2l+1}{18}\sum_{\vec{k}} \Phi^2(k)\left(\frac{a}{kR_{\rm rec}}\right)^2. \tag{7}$$

Therefore, amplitudes of low multipoles are much smaller (at least, $aL_1/R_{\rm hor}$ times) than $\Delta T/T$ fluctuations at angles $\vartheta \sim aL_1/R_{\rm hor}$ (in radians).[2)]

Multipole dependence of the observed large-scale $\Delta T/T$ fluctuations[9] does not follow this relation. Actually, it is much better fitted by the law $(\Delta T/T)_l^2 \propto [2l+1/l(l+1)]$ which follows from the inflationary scenario.[13] Thus, a sufficiently small 3-torus Universe is excluded by the data. Let us find more exact restrictions on the model. Another possible fit to the angular correlation function $\zeta_T(\vartheta)$ of the COBE data is a Gaussian with the correlation angle $\vartheta_c = 13.5 \pm 2.5^\circ$ (see Ref. 14). This corresponds to $l_m = 3$–5, where l_m is the multipole with the largest amplitude $(\Delta T/T)_l$. We can assume, therefore, that $l_m \leqslant 6$.

First, consider the modes with $npq \neq 0$ having a generic dependence on $\vec{r}$ and (θ,φ). For them $k > 2\pi/L_3$. The quantity $(2l+1)j_l^2(v)$ considered as a function of integer l has a maximum at $l = l_m \leqslant 6$ for $v < 8.5$. Thus, a lower limit on the minimal topological size of the Universe follows:

$$aL_3 > 0.75R_{\rm hor} \approx 9000 \text{ Mpc}. \tag{8}$$

Second, from modes with only one of the numbers n. p, q equal to zero, the modes with $q=0$, $n^2+p^2 \neq 0$ are most important. If the contribution to $\Delta T/T$ from these modes dominates at low l, then L_3 may be much less than $R_{\rm hor}/a$ and the estimate (8) refers to the intermediate size aL_2. However, in this case the large-scale $\Delta T/T$ pattern has a symmetry plane because of the absence of the dependence of the corresponding part of $\Phi(\vec{r})$ on z; i.e.,

$$\frac{\Delta T}{T}(\theta,\varphi) = \frac{\Delta T}{T}(\pi - \theta,\varphi) \tag{9}$$

in a system of angular coordinates. For $aL_2 \gg R_{\rm hor}$, the observed part of the Universe has the effective spatial topology T^1.

Finally, from modes with two of the numbers n, p, q equal to zero, the modes with $p=q=0$, $n \neq 0$ are the most interesting modes. The may give the main contribution to $\Delta T/T$ at low l if L_2 and L_3 are much less than $R_{\rm hor}/a$. Then the lower limit (8) refers to the maximal size aL_1 only, but the large-scale $\Delta T/T$ pattern has a symmetry axis:

$$\frac{\Delta T}{T}(\theta,\varphi) = \frac{\Delta T}{T}(\theta). \tag{10}$$

For $aL_1 \gg R_{\rm hor}$ we come to the case of the effective T^2 spatial topology.

Therefore, three possibilities exist: 1) all sizes of a 3-torus satisfy estimate (8); 2) estimate (8) refers to the two larger sizes, but the large-scale $\Delta T/T$ pattern has a symmetry plane (9); 3) the maximal size of the torus alone satisfies (8), and Eq. (10)

is valid. The $\Delta T/T$ map obtained in Ref. 9 is noisy, so it does not immediately provide the real picture of primordial fluctuations. Nevertheless, Eq. (10) imposes so much symmetry that the third case can probably be excluded. The symmetry imposed by Eq. (9) in the second case is less evident visually, so more accurate maps are required to exclude this case definitely. If this goal is achieved, we would be sure that all topological sizes are at least on the order of the present cosmological horizon, in accordance with the prediction of the inflationary scenario. The $\Delta T/T$ fluctuations produced by primordial gravitational waves are similar to those produced by adiabatic perturbations (although they depend on the values of gravitational waves between $t_{\rm rec}$ and t_0 at all times). The same conclusions are therefore expected to be valid in this case.

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[1)]The results of this paper were presented in the author's plenary talk at the Texas/PASCOS'92 Conference (Berkeley, USA, Dec. 13–18, 1992). The author became aware of the recent papers,[10] where similar results for the $L_1 = L_2 = L_3$ case were obtained independently.

[2)]Here the author would like to rectify an incorrect statement that $(\Delta T/T)_2 = 0$ in the case $L_1 = L_2 = L_3$ that appears in the beginning of Ref. 6 (fortunately, it was not used in any way in the rest of the paper). The correct statement is that this quantity is small (much smaller than 10^{-5}) if $aL_1 \ll R_{\rm hor}$.

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