Motion of a 2D electron along the edge of a sample in a high magnetic field

S. V. Iordanskiĭ and B. A. Muzykantskiĭ
L. D. Landau Institute of Theoretical Physics, Russian Academy of Sciences, 117334 Moscow, Russia

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The motion of a 2D electron near the edge of a sample in a high magnetic field is analyzed. Scattering by impurities near the edge is taken into account. It is assumed that the correlation length of the potential set up by the impurities is large. The electron drift velocity is shown to be inversely proportional to the number of 1D channels which form near the edge.

The edge states of 2D electrons in high magnetic fields are currently the subject of active theoretical and experimental research. In particular, much progress is being made in research on plasmon edge modes under conditions of the quantum Hall effect, such that the spectrum of edge excitations is shaped by the Coulomb interaction between electrons.[1,2] To deal simultaneously with the electron–electron interaction and scattering by an impurity potential is a difficult problem, which has yet to be solved.[3] In this situation it seems worthwhile to examine the effect of an impurity potential on the properties of edge states in the absence of a Coulomb interaction.

We consider a potential which is smooth at the scale of the magnetic length. In the very simple drift approximation, an electron moves along contour lines $V(x,y) = E$, and the wave function is

$$\chi_E(x,y) = e^{i\varphi(s)} f_0(n/l_h), \quad \varphi(s) = \frac{e}{c}\int^s \mathbf{A}_h ds, \tag{1}$$

where $\mathbf{A}_h$ is the vector potential of the external magnetic field, f_0 is an oscillator wave function, l_h is the magnetic length, n is the distance to the contour line along the normal, and s is the distance along the contour line.

Figure 1a is a sketch of the potential and the contour lines near the edge of the sample. The normal to the edge intersects any contour line $V(x,y) = E$ an odd number of times. The electron drift velocities at the points of intersection alternate in accordance with the sign of $\partial V/\partial n$. Wave functions (1) on different components of a contour line overlap near the saddle points of the potential. As a result, an electron may tunnel from the edge to nearby "islands" circled by a contour line with the same energy. The electron drift velocity along an edge decreases as a result.

We show below that this circumstance does not result in a localization of electron states. We derive a renormalized electron drift velocity along an edge.

We consider the model situation in which each level $V = E$ is repeated no more than three times (Fig. 1b; the crosses are saddle points of the potential). The normal

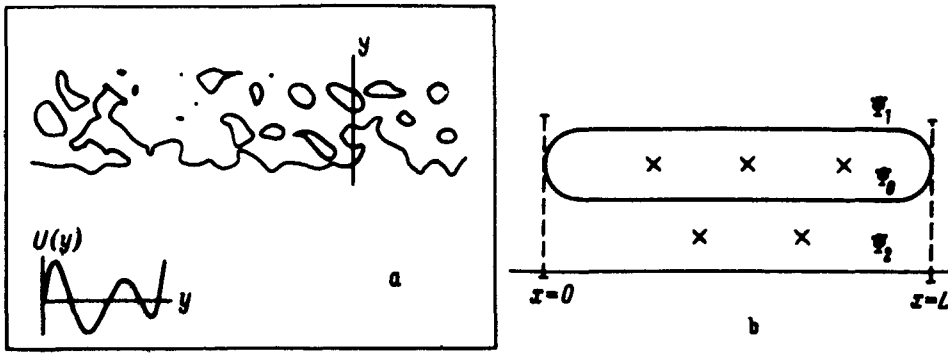

FIG. 1. *a*—Contour lines near the edge of the sample and profile of the potential along the normal to the edge; *b*—model used to clarify the behavior of the electron wave function. The crosses are saddle points of the potential.

to the edge at a typical point intersects the contour line $V(x,y)=E$ three times, so we must treat the wave function as a three-component complex vector

$$\Psi(x)=\begin{pmatrix}\Psi_0\\ \Psi_1\\ \Psi_2\end{pmatrix}.$$

We assume that the electron moves to the left in the zeroth channel and to the right in the first and second. We first consider the case in which the velocities in all channels are equal in magnitude: $v_0=v_1=v_2\equiv v_d$.

The transfer matrix which relates the values of the wave function at the ends of a disordered section, $\Psi(x)=\hat{T}(x)\Psi(0)$, must conserve the total current through the three channels:

$$j=v_d(-|\Psi_0|^2+|\Psi_1|^2+|\Psi_2|^2)=v_d\langle\Psi\Psi\rangle. \tag{2}$$

The pseudoscalar product introduced in (2) is thus conserved by the transfer matrix:

$$\langle AB\rangle=A^+\hat{g}B,\quad g=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \langle\hat{T}A\hat{T}B\rangle=\langle AB\rangle. \tag{3}$$

We consider the Born weak-scattering case, assuming that the increments in the transfer matrix with increasing length of the disordered segment are independent random quantities of a white-noise sort:

$$\frac{d\Psi}{dx}=\hat{\Xi}(x)\Psi,\quad \hat{\Xi}=i\begin{pmatrix}\gamma_0&-\delta_1&-\delta_2\\ \delta_1&\gamma_1&0\\ \delta_2&0&\gamma_2\end{pmatrix}, \tag{4}$$

where γ_0, γ_j, and δ_j $(j=1,2)$ are random real functions of the coordinate, and $\overline{[\delta_k(x)\delta_j(x')]}=(1/2l)\delta_{k,j}\delta(x-x')$. The symmetry of Ξ follows from (3). The zero matrix elements reflect the circumstance that there is no scattering from the first channel into the second. The diagonal elements of $\hat{\Xi}$ are large in comparison with the off-diagonal ones, since they include a systematic change in the phase of wave functions (1) in the absence of scattering.

We are interested in the time it takes an electron to traverse a disordered section of length L of the edge. The boundary conditions can naturally be taken to be

$$\Psi_0(0)=\Psi_1(0), \quad \Psi_0(L)=\Psi_1(L), \quad \Psi_2(0)=1, \tag{5}$$

corresponding to a closure of the zeroth and first channels in a loop (Fig. 1b). By virtue of current conservation, the change in the wave function of an electron incident through the second channel reduces to a phase shift $\Psi_2(L)=e^{i\theta}$. We now assume that a wave packet $\Psi_{2,in}=\int f(p)e^{ipx}dp$ is incident on the disordered section from the left in the second channel. The wave function at the exit end after a time t is

$$\Psi_{2,\text{out}}=\int f(p)e^{-iEt+i\theta(E,L)}dp, \tag{6}$$

where $E=pv_d$ is the energy of the electron. It is easy to see that a packet $f(p)=\text{const}$, which is narrow in coordinate space, appears at the opposite end after a time delay $\tau=\partial\theta/\partial E$. On the other hand, the quantity $\partial\theta/\partial E$ is simply the density of states in the disordered section. It is not renormalized when scattering between channels is taken into account, so we have $\overline{(\tau)}=(3/v_d)L$. This result can immediately be generalized to the multichannel case in which an electron moves in one direction in n channels and in the other direction in $n+1$ channels:

$$\overline{(\tau)}=L\sum_{i=1}^{2n+1}\frac{1}{v_i}, \tag{7}$$

where v_i are the magnitudes of the velocities in the channels.

An interpretation of the density of states as the packet traversal time is legitimate if the fluctuations in τ are small: $\Delta\tau\equiv\sqrt{\overline{(\tau-\bar{\tau})^2}}\ll\tau$. We can show that we have $\Delta\tau\sim\sqrt{L}$ for disordered sections which are sufficiently long $(L\gg l)$.

It is convenient to relate the time required to traverse a disordered section to the average current:

$$\frac{1}{\tau}=\frac{j}{\Sigma_i\int_0^L L|\Psi_i|^2dx}. \tag{8}$$

To evaluate the normalization integral we need to transfer boundary conditions (5) to an internal point. We consider the auxiliary vectors $\xi^{\pm}(x)$, which satisfy the equation $d\xi^{\pm}/dx=\hat{\Xi}\xi^{\pm}$ and the boundary conditions

$$\xi^{+}(0)=\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \xi^{-}(L)=\begin{pmatrix}1\\1\\0\end{pmatrix}, \tag{9}$$

which are imposed at respectively the left and right boundaries of the disordered section. The vectors $\xi^{\pm}(x)$ are orthogonal to $\Psi(x)$ in the sense of scalar product (3):

$$0=\langle\xi^{+}(0)\Psi(0)\rangle=\langle\xi^{-}(L)\Psi(L)\rangle=\langle\xi(x)\Psi(x)\rangle. \tag{10}$$

The first two of these equalities follow from (5) and (9); the third follows from the invariance of product (3). Since we have $\langle\Psi(x)\Psi(x)\rangle=\langle\Psi(0)\Psi(0)\rangle=1$, the directions of the vectors $\xi^{\pm}(x)$ determine $\Psi^{\pm}(x)$ within a phase factor. In particular, the square magnitude of the wave function is

$$|\Psi(x)|^2\equiv\Sigma|\Psi_i|^2\frac{|\xi_1^+-\xi_1^-|^2+|\xi_2^+-\xi_2^-|^2+|\Delta|^2}{|\xi_1^+-\xi_1^-|^2+|\xi_2^+-\xi_2^-|^2-|\Delta|^2},\quad \Delta=\xi_2^+\xi_1^--\xi_2^-\xi_1^+. \tag{11}$$

The quantities $\xi^{\pm}(x)$ are statistically independent, since they are determined by transfer matrices on nonintersecting sections $[0,x]$ and $[x,L]$, respectively. We introduce the coordinates

$$\xi^{+}=\Lambda\begin{pmatrix}1\\ \cos\phi e^{i\alpha_1}\\ \sin\phi e^{i\alpha_2}\end{pmatrix}, \tag{12}$$

where we have used $\langle\xi^{+}(x)\xi^{+}(x)\rangle=0$. The set of directions of the light vectors (i.e., of vectors with a zero scalar square) are parametrized by the points of the 3D sphere S^3. The dynamics in (4) thus induces a random walk on S^3 if the direction of the light vector ξ^{+} alone is followed. Since the diagonal elements of $\hat{\Xi}$ are large in comparison with the off-diagonal elements, the phases α_1 and α_2 are fast variables, over which we can take an average.[4] The remaining Langevin equation for ϕ is

$$\frac{d\phi}{dx}=-\delta_1\sin\alpha_1\sin\phi+\delta_2\sin\alpha_2\cos\phi+\frac{1}{2l}(\cot\phi-\tan\phi). \tag{13}$$

The corresponding Fokker–Planck equation,

$$2l\frac{\partial P(\phi,L)}{\partial L}=\frac{\partial}{\partial\phi}\left(-2\cot 2\phi P+\frac{\partial P}{\partial\phi}\right), \tag{14}$$

has a stable, steady-state solution $P_{\infty}=\sin(2\phi)$ which (as is easily shown) is the same as a uniform distribution over the sphere S^3. The same is true of the direction of the vector ξ^{-}. Using these results for the average in (11), we easily find $\overline{(|\Psi(x)|^2)}=3$. Also using (8), we again find (7) for $n=1$.

The estimate of $\Delta\tau$ which we are seeking follows from the existence of a steady-state distribution for the directions of the vectors $\xi^{\pm}$. To ensure that a steady-state distribution is established in each segment, it is sufficient to break up the integral

$$\int_0^L\sum_i|\Psi_i|^2dx$$

in the denominator of (8) into segments of length ΔL, $l\ll\Delta L\ll L$. The integrals over the segments will be statistically independent, leading to the estimate of the dispersion $\Delta\tau$ given above.

We have thus shown that the average time taken to traverse a disordered section of the edge is equal to the density of edge states in (7) and is inversely proportional to the number of 1D channels which form near the edge. The traversal time for a characteristic fluctuation has been estimated to be $\Delta\tau \sim \sqrt{L} \ll \tau$.

While working on this paper, we had the opportunity to see the results of a recent paper by Barnes *et al.*,[5] who discuss the transmission of a system of 1D channels in each of which an electron moves in one direction. There is a scattering between channels in their problem. We would like to point out that the results of Ref. 5 agree with our calculations and are essentially a consequence of current conservation. To show this, we note that the transfer matrix $\hat{T}$ in a system with M rightward and N leftward channels (for definiteness, we assume $M>N$) belongs to the group SU (M,N); i.e., it conserves the form

$$\langle \mathbf{AB} \rangle = \mathbf{A}^{+}\hat{g}_{M,N}\mathbf{B}, \quad \hat{g}_{M,N} = \mathrm{Diag}(\overbrace{1,\ldots,1}^{M} \quad \overbrace{-1,\ldots,-1}^{N}), \quad \hat{T}^{+}\hat{g}_{M,N}\hat{T} = \hat{g}_{M,N}. \quad (15)$$

It can be seen from the last equality, $\hat{g}_{M,N}^{-1}\hat{T}^{+}\hat{g}_{M,N} = \hat{T}^{-1}$, that the eigenvalues $\{\lambda_1^*,\ldots,\lambda_{M,N}^*\}$ and $\{\lambda_1^{-1},\ldots,\lambda_{N+M}^{-1}\}$ of the matrices $\hat{T}^{+}$ and $\hat{T}^{-1}$ are the same. Consequently, either we have $|\lambda_i| = 1$ or there exists yet another eigenvalue $\lambda_j, \lambda_i = 1/\lambda_j^*$. It is a straightforward matter to show, by orthogonalizing the eigenbasis of $\hat{T}$ and using the law of inertia of quadratic forms, that the number (K) of eigenvalues of $\hat{T}$, which are equal to one in magnitude, is no less than $M-N$. At least $M-N$ of the corresponding eigenvectors correspond to states which are propagating to the right. An important point is that the strict inequality $K>M-N$ holds only for matrices which are sufficiently close to unity.[1)] In a sufficiently long system of scatterers there are thus precisely $N-M$ states propagating to the right, while all states propagating in the other direction exponentially grow or decay and are thus localized.

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[1)]This assertion can be proved easily for the very simple SU(1,1) case, to which the general case can be reduced.

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