# Deformation of some integrable equations

T. A. Ivanova[1)] and A. D. Popov[2)]
*Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow Region, Russia*

(Submitted 8 December 1994)
Pis'ma Zh. Eksp. Teor. Fiz. **61**, No. 2, 142–148 (25 January 1995)

It is shown that deformations of integrable equations in $1 \leqslant d \leqslant 3$ or, in other words, nonautonomous versions of well-known integrable equations can be obtained by reduction of the self-duality equations of the Yang–Mills model in $d=4$ under the action of symmetry groups. New nonautonomous integrable equations (and their linear systems), which are deformations of the equation of the principal chiral model in $d=3$, of the Korteweg–de Vries equation, and of the equations of the Hamiltonian systems with quartic potentials, are described. © *1995 American Institute of Physics.*

**1.** Most of the integrable equations in $(1+1)$ dimensions arise as compatibility conditions of the overdetermined linear system of equations[1,2]

$$(\partial_t+V)\Psi=0, \quad (\partial_x+U)\Psi=0. \tag{1}$$

Here $V$ and $U$ are matrices from the algebra $gl(n,C)$ which depend on the coordinates $t$, $x$ and on a constant complex spectral parameter $\lambda$, and $\Psi \in C^n$ is a vector-function which depends on $t$, $x$ and $\lambda$. It was shown,[3,4] however, that the Ernst equation arises as a compatibility condition of a linear system which is more general than (1). Belinsky and Zakharov have generalized (1) by adding terms with a derivative with respect to the spectral parameter[3] $\lambda$. They have also generalized the inverse scattering transform method for solving that type of systems. At the same time, Maison wrote the linear system for the Ernst equation in the form of (1) but with the spectral parameter $\lambda$, which is a function of $t$ and $x$ and of a hidden spectral parameter.[4] The dressing method for this linear system has been developed by Mikhailov and Yaremchuk.[5] They also constructed the explicit solutions and investigated the conservation laws. Later on, Burtsev and coworkers have shown that the two generalizations of the linear system (1) are equivalent and developed the general method of integration of nonlinear equations that arise as compatibility conditions of such generalized linear systems. Moreover, they have shown that with the help of the above-described generalization of the linear system (1) to each equation in $(1+1)$ dimensions, integrable by the inverse scattering transform method, we can associated some integrable equations with the coefficients that depend on $t$ and $x$. They called these equations the deformations of initial equations.

On the other hand, Witten[7] showed (see also Ref. 8) that the Ernst equation can be obtained by reducing the self-dual Yang–Mills (SDYM) equations which were introduced in the landmark paper of Belavin *et al.*[9] The integrability of the SDYM equations was proved by Belavin and Zakharov[10] and Ward.[11] Now it is known[12,8] that the reduction of the SDYM equations under two translations yields an equation of the principal chiral model in $d=2$, and the reduction of the SDYM equations under translation and rotation yields a deformed equation of the principal chiral model which, in the particular case of the gauge group $GL(2,R)$, is equivalent to the Ernst equation.

Later, it was shown that not only the equation of the principal chiral model in $d=2$, but also many other integrable equations in $1\leq d\leq 3$ dimensions may be obtained by reductions of the SDYM equations with respect to the action of subgroups of the group of translations in $R^4$ (see, e.g., Refs. 12–14 and the references cited there). It is natural to assume that the replacement of some generations of the translation group by generators of the rotation group will permit us to obtain, by symmetry reduction of the SDYM equations, the deformation not only of the equation of the principal chiral model in $d=2$, but also of some other integrable equations. The purpose of this letter is to prove this assumption.

We show that the deformed nonlinear Schrödinger equation (NLS), which was considered in Refs. 6 and 15, can be deduced by reducing the SDYM equations. We also describe three new examples of nontrivial deformations of the well-known integrable equations (and their linear systems), which can be obtained by reductions of the SDYM equations under different symmetry groups. Among them there are deformations of the equation of the principal chiral model in $R^{1,2}$, which was considered by Manakov and Zakharov,[16] the Korteweg–de Vries (KdV) equation, and the equations of the Hamiltonian systems with quartic potentials.

**2.** We consider the space $R^{2,2}$ with the metric $(g_{\mu\nu})=\mathrm{diag}(+1,+1,-1,-1)$ and the potentials $A_\mu$ of the Yang–Mills (YM) fields $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$, where $\mu,\nu,...=1,...,4$, $\partial_\mu=\partial/\partial x^\mu$. The fields $A_\mu$ and $F_{\mu\nu}$ take values in the Lie algebra $gl(n,C)$.

In $R^{2,2}$ we introduce the null coordinates

$$t=\tfrac{1}{2}(x^2-x^4),\quad u=\tfrac{1}{2}(x^2+x^4),\quad y=\tfrac{1}{2}(x^1-x^3),\quad z=\tfrac{1}{2}(x^1+x^3)$$

and set $A_t=A_2-A_4$, $A_u=A_2+A_4$, $A_y=A_1-A_3$, and $A_x=A_1+A_3$. The SDYM equations in the null coordinates have the form

$$F_{tz}=0,\quad F_{uy}=0,\quad F_{tu}+F_{zy}=0. \tag{2}$$

Equations (2) can be obtained as compatibility conditions of the following linear system of equations (cf. Refs. 10 and 11):

$$(\partial_t-\lambda\partial_y+A_t-\lambda A_y)\Psi=0, \tag{3a}$$

$$(\partial_z+\lambda\partial_u+A_z+\lambda A_u)\Psi=0, \tag{3b}$$

$$\partial_{\bar\lambda}\Psi=0, \tag{3c}$$

where $\bar\lambda$ is the complex conjugate to $\lambda$. Here $\Psi$ is a column vector which depends on the coordinates of $R^{2,2}$ and on the "coordinates" $\lambda$ and $\bar\lambda$, and which parametrizes the upper sheet of the hyperboloid $H^2=SO(2,1)/SO(2)$. Notice that $\Psi$ is defined on the twistor $\mathscr{Z}=R^{2,2}\times H^2$ for the space $R^{2,2}$ and Eqs. (3) mean the holomorphicity of the vector-function $\Psi$ (Ward theorem[11]).

**3.** We consider the inhomogeneous group of rotations $ISO(2,2)$ (rotations and translations) and an arbitrary subgroup $G$ of the group $ISO(2,2)$. We would like to impose the conditions of $G$-invariance on the potentials $A_\mu$ and on the vector-function $\Psi$. For this purpose we must define the generators of the group $ISO(2,2)$ as vector fields on $R^{2,2}$, when considering the action of $G$ on $A_\mu$, and as vector fields on the twistor space $R^{2,2}\times H^2$, when considering the action of $G$ on $\Psi$ (Ref. 17).

We introduce the following constant tensors:

$$f^a_{\mu\nu}=\{f^a_{bc},\ \mu=a,\ \nu=b;\ \delta^a_\mu,\nu=4;\ -\delta^a_\nu,\mu=4\},\quad I^{\ \mu}_{a\nu}=-\frac{1}{2}g_{ab}g^{\mu\lambda}f^b_{\lambda\nu},\tag{4a}$$

$$\bar f^a_{\mu\nu}=\{f^a_{bc},\ \mu=a,\nu=b;\ -\delta^a_\mu,\nu=4;\ \delta^a_\nu,\mu=4\},\quad J^{\ \mu}_{a\nu}=-\frac{1}{2}g_{ab}g^{\mu\lambda}\bar f^b_{\lambda\nu},\tag{4b}$$

where $a,b,...=1,2,3$, $g_{11}=g_{22}=-g_{33}=1$, and $f^1_{23}=f^2_{31}=-f^3_{12}=1$ are the structure constants of the group $SO(2,1)$. The generators of the group $ISO(2,2)$ can then be realized in terms of the following vector fields on $R^{2,2}$:

$$X_a=I^{\ \mu}_{a\nu}x^\nu\partial_\mu,\quad Y_a=J^{\ \mu}_{a\nu}x^\nu\partial_\mu,\quad P_\mu=\partial_\mu.\tag{5}$$

The vector fields on $\mathscr{Z}=R^{2,2}\times H^2$, which also form the generators of $ISO(2,2)$, are given by

$$\tilde X_a=X_a,\quad \tilde Y_a=Y_a+Z_a,\quad \tilde P_\mu=P_\mu,\tag{6a}$$

with the following expression of the generators $Z_a$ of the $SO(2,1)$ rotations on $H^2$

$$Z_1=\frac{1}{2}[(1-\lambda^2)\partial_\lambda+(1-\bar\lambda^2)\partial_{\bar\lambda}],\quad Z_2=-[\lambda\partial_\lambda+\bar\lambda\partial_{\bar\lambda}],$$

$$Z_3=-\frac{1}{2}[(1+\lambda^2)\partial_\lambda+(1+\bar\lambda^2)\partial_{\bar\lambda}].\tag{6b}$$

It can be easily shown that $[X_a,X_b]=f^c_{ab}X_c$, $[Z_a,Z_b]=f^c_{ab}Z_c$, etc.

To reduce the SDYM equations (2) and the linear system (3) under a subgroup $G$ of the group $ISO(2,2)$ it is necessary to impose the following conditions of $G$ invariance on the potentials $A_\mu$ and on the vector-function[18] $\Psi$:

$$W_\xi A_\mu+A_\sigma W^\sigma_{\xi,\mu}=0,\quad \forall\xi\in\mathscr{G},\tag{7a}$$

$$\tilde W_\xi\Psi=0,\quad \forall\xi\in\mathscr{G},\tag{7b}$$

where $\mathscr{G}$ is the Lie algebra of the group $G$, $W_\xi=W^\sigma_\xi\partial_\sigma$ are the vector fields on $R^{2,2}$, and $\tilde W_\xi$ are the vector fields on $R^{2,2}\times H^2$. Both $W_\xi$ and $\tilde W_\xi$ form a realization of the Lie algebra $\mathscr{G}$.

In accordance with the general method of symmetry reduction (see Ref. 19, and the references cited there), as new coordinates on $R^{2,2}$, we should choose the coordinates $\theta_\xi$ on the orbits of the group $G$, and the invariant coordinates $\theta_A$ $(A=1,...,4-\dim G)$ and $\zeta$ which parametrize the space of orbits and satisfy the relations

$$\tilde W_\xi\theta_A=0,\quad \tilde W_\xi\zeta=0,\quad \partial_{\bar\lambda}\zeta=0,\quad \forall\xi\in\mathscr{G}.\tag{8}$$

Here the invariant complex coordinate $\zeta$ represents the new "spectral parameter." Substituting solutions of Eqs. (7) and (8) into Eqs. (2) and (3), we obtain the reduced SDYM equations and their linear system in terms of the functions of the invariant coordinates.[17,19]

**4.** *Deformed chiral model equation in $R^{2,1}$*. We consider the one-dimensional group of rotations generated by the vector field $X_2+Y_2$. From (5) and (6) we obtain $\tilde{X}_2+\tilde{Y}_2=X_2+Y_2+Z_2=z\partial_z-y\partial_y-\lambda\partial_\lambda - \bar{\lambda}\partial_{\bar{\lambda}}$. On $R^{2,2}\times H^2$ let us introduce the coordinates $\rho$, $\theta$, $\eta$, and $\xi$ by the formulas $y = \frac{1}{2}\rho e^{-\theta}$, $z = \frac{1}{2}\rho e^{\theta}$, and $\lambda=\eta e^{i\xi}$. We then will have $X_2+Y_2=\partial_\theta$ and $\tilde{X}_2+\tilde{Y}_2=\partial_\theta-\eta\partial_\eta$. Therefore, $\varphi = \frac{1}{2}121212121212121212(\theta - \ln\eta)$ will be the coordinate on the orbit and $t,u,\rho,\zeta=\lambda e^{\theta}$ will be the invariant coordinates.

The invariant YM potentials $A_\mu$, which satisfy Eqs. (7a), have the form

$$A_t=T_t(t,u,\rho),\quad A_u=T_u(t,u,p),\quad A_y=T_y(t,u,\rho)e^{\theta},\quad A_z=T_z(t,u,\rho)e^{-\theta}. \tag{9a}$$

The vector-function

$$\Psi=\psi(t,u,\rho,\zeta) \tag{9b}$$

is the solution of Eqs. (7b) and (3c).

Substituting (9) into (3), we obtain the following reduced linear system:

$$\nabla_{V_1}\psi\equiv\left[\partial_t-\zeta\partial_\rho+\frac{1}{\rho}\zeta^2\partial_\zeta+T_t-\zeta T_y\right]\psi=0,$$

$$\nabla_{V_2}\psi\equiv\left[\partial_\rho+\zeta\partial_u+\frac{1}{\rho}\zeta\partial_\zeta+T_z+\zeta T_u\right]\psi=0, \tag{10}$$

where $V_1 = \partial_t - \zeta\partial_\rho + \frac{1}{\rho}\zeta^2\partial_\zeta$, and $V_2 = \partial_\rho + \zeta\partial_u + \frac{1}{\rho}\zeta\partial_\zeta$. Recall that, in general, we have $[V_1,V_2]\neq 0$, and then for linear systems like (10) the compatibility condition is

$$[\nabla_{V_1},\nabla_{V_2}]-\nabla_{[V_1,V_2]}=0. \tag{11}$$

Correspondingly, the SDYM equations (2) are reduced to

$$\partial_t T_z-\partial_\rho T_t+[T_t,T_z]=0,\quad \partial_\rho T_u-\partial_u T_y+[T_y,T_u]=0, \tag{12a}$$

$$\partial_\rho(T_y-T_z)+\frac{1}{\rho}(T_y-T_z)+\partial_t T_u-\partial_u T_t+[T_t,T_u]+[T_z,T_y]=0, \tag{12b}$$

consistent with the compatibility condition (11) of the linear system (10).

Choosing the algebraic constraints $T_z=T_t=0$, from Eqs. (12a) we obtain $T_u=g^{-1}\partial_u g$ and $T_y=g^{-1}\partial_\rho g$. Equation (12b) then reduces to the deformed chiral model equation in $R^{2,1}$

$$\partial_\rho(g^{-1}\partial_\rho g)+\frac{1}{\rho}g^{-1}\partial_\rho g+\partial_t(g^{-1}\partial_u g)=0 \Leftrightarrow \frac{1}{\rho}\partial_\rho(\rho g^{-1}\partial_\rho g)+\partial_t(g^{-1}\partial_u g)=0. \tag{13}$$

The linear system for this equation has the form (10) with $T_z=T_t=0$.

*Remark.* Notice that if one uses an additional condition of invariance under $P_t+P_u$: $(\partial_t+\partial_u)\psi=(\partial_t+\partial_u)g=0$ in the linear system (10) and in Eq. (13), then one obtains the deformed equation of the principal chiral model in $R^{1,1}$ (Ref. 6), which in the particular case of the gauge group $GL(2,R)$ is equivalent to the Ernst equation.[8]

**5.** Here we consider the examples of reductions of the SDYM equations to the integrable equations in $R^{1,1}$.

*Deformed NLS equation.* Let us consider the two-dimensional Abelian group with the generators $\{X_2+Y_2,P_u\}$. The invariant $A_\mu$ and $\Psi$ are then given by Eqs. (9), where $T_\mu$ and $\psi$ do not depend on $u$. The reduced linear system and reduced SDYM equations have the form

$$\left[\partial_t-\zeta\partial_\rho+\frac{1}{\rho}\,\zeta^2\partial_\zeta+T_t-\zeta T_y\right]\psi=0,\quad \left[\partial_\rho+\frac{1}{\rho}\,\zeta\partial_\zeta+T_z+\zeta T_u\right]\psi=0, \tag{14}$$

$$\partial_t T_z-\partial_\rho T_t+[T_t,T_z]=0,\quad \partial_\rho T_u+[T_y,T_u]=0, \tag{15a}$$

$$\partial_\rho(T_y-T_z)+\frac{1}{\rho}(T_y-T_z)+\partial_t T_u+[T_t,T_u]+[T_z,T_y]=0. \tag{15b}$$

For matrices from (14) and (15) we choose the ansatz

$$T_t=\begin{pmatrix} a & \bar{b} \\ b & -a\end{pmatrix},\quad T_u=\frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix},\quad T_y=0,\quad T_z=\sqrt{\kappa}\begin{pmatrix} 0 & \bar{\phi} \\ \phi & 0\end{pmatrix}, \tag{16}$$

where $a$, $b$ and $\phi$ are arbitrary complex-valued functions of $t$ and $\rho$, $\kappa$ is an arbitrary, real constant parameter, and the bar over the letter means complex conjugation. Substituting (16) into (15), we find that

$$a=-i\kappa\bar{\phi}\phi-2i\kappa\int\frac{d\rho}{\rho}\bar{\phi}\phi,\quad b=i\sqrt{\kappa}\left(\partial_\rho\phi+\frac{1}{\rho}\,\phi\right), \tag{17}$$

and that the function $\phi$ must satisfy the equation

$$i\partial_t\phi+\partial_\rho^2\phi-2\kappa(\bar{\phi}\phi)\phi=-\frac{1}{\rho}\,\partial_\rho\phi+\frac{1}{\rho^2}\,\phi+4\kappa\phi\left(\int\frac{d\rho}{\rho}\bar{\phi}\phi\right). \tag{18}$$

The linear system for Eq. (18) can be derived by substituting (16) and (17) into (14).

*Remark.* The deformed NLS equation (18) was considered in Ref. 6. When $\kappa=-1$, this equation is gauge equivalent to the equation of the Heisenberg ferromagnet in axial geometry. By changing the variables $t$, $\rho$, and $\phi$, Eq. (18) can be transformed to the equation which was introduced and integrated in Ref. 15. Thus, the deformed NLS equation is shown to be the reduction of the SDYM equations.

*Deformed KdV equation.* Let us now consider the generators $\{X_2+Y_2,P_u\}$, the linear system (14), and its compatibility conditions (15). For the matrices from (15) we choose the ansatz

$$T_t=\begin{pmatrix} a & b \\ c & -a\end{pmatrix},\quad T_u=\begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix},\quad T_y=\begin{pmatrix} 0 & 0 \\ h & 0\end{pmatrix},\quad T_z=\begin{pmatrix} 0 & g \\ f & 0\end{pmatrix}, \tag{19}$$

where $a$, $b$, $c$, $f$, $g$, and $h$ are the arbitrary real-valued functions of $t$ and $\rho$. Substituting (19) into (15), we find that

$$a=\frac{1}{4}\,\partial_\rho f-\frac{1}{4\rho}\int\frac{d\rho}{\rho}f,\quad b=-\frac{1}{2\rho}f-\frac{1}{2\rho}\int\frac{d\rho}{\rho}f,\quad h=\frac{1}{2}f+\frac{1}{2}\int\frac{d\rho}{\rho}f,$$

$$c=\frac{1}{4}\,\rho\partial_\rho^2 f-\frac{1}{4\rho}f+\frac{1}{2}f^2+\left(\frac{1}{4\rho}+\frac{f}{2}\right)\int\frac{d\rho}{\rho}f,\quad g=-\frac{1}{\rho},\tag{20}$$

and that the function $f$ satisfies the equation

$$\partial_t f-\frac{3}{2}f\partial_\rho f-\frac{1}{4}\,\partial_\rho^3(\rho f)$$

$$=\frac{1}{2\rho^2}f+\frac{1}{2\rho}f^2-\frac{1}{4\rho}\,\partial_\rho f-\frac{1}{2}\,\partial_\rho^2 f+\left(\frac{1}{2}\,\partial_\rho f-\frac{f}{2\rho}-\frac{1}{4\rho^2}\right)\int\frac{d\rho}{\rho}f.\tag{21}$$

The linear system for Eq. (21) is deduced by inserting (19) and (20) into the linear system (14).

*Remark.* The deformed KdV equations were analyzed in Refs. 6 and 15. Equation (21) differs from those considered in Refs. 6 and 15 and it is a new deformation of the KdV equation.

**6.** Finally, we consider the reduction of the SDYM equations to those of the Hamiltonian systems with quartic potentials.

*Deformed equations of the Hamiltonian systems with quartic potentials.* Let us consider the three-dimensional Abelian subgroup of $ISO(2,2)$ generated by the vector fields $X_2+Y_2,P_u,P_t$. It is easy to show that the invariant YM potentials and $\Psi$, which satisfy Eqs. (3c) and (7b), are given by Eqs. (9), where $T_\mu$ and $\psi$ depend only on $\rho$ and do not depend on $t$ and $u$. Substituting the invariants $A_\mu$ and $\psi$ into the linear system (3) and using (7), we obtain the following reduced linear system:

$$\left[\zeta\partial_\rho-\frac{1}{\rho}\,\zeta^2\partial_\zeta-T_t+\zeta T_y\right]\psi=0,\quad \left[\partial_\rho+\frac{1}{\rho}\,\zeta\partial_\zeta+T_z+\zeta T_u\right]\psi=0.\tag{22}$$

Using the compatibility condition (11) for the linear system (22), we obtain the following reduced SDYM equations:

$$\frac{d}{d\rho}\,T_t+[T_z,T_t]=0,\quad \frac{d}{d\rho}\,T_u+[T_y,T_u]=0,\tag{23a}$$

$$\frac{d}{d\rho}(T_y-T_z)+\frac{1}{\rho}(T_y-T_z)+[T_t,T_u]+[T_z,T_y]=0.\tag{23b}$$

Let us choose in $GL(n,C)$ the subgroups $N$ and $H$, so that $N/H$ be a compact Hermitian symmetric space [for example, $N=SU(n),H=SU(n-m)\times SU(m)\times U(1)$].[20] Let $\mathcal{N}$ and $\mathcal{H}$ be the Lie algebras of the Lie groups $N$ and $H$. Then $\mathcal{N}=\mathcal{H}\oplus\mathcal{P}$ and $[\mathcal{H},\mathcal{H}]\subset\mathcal{H},[\mathcal{H},\mathcal{P}]\subset\mathcal{P},[\mathcal{P},\mathcal{P}]\subset\mathcal{H}$. A special feature of Hermitian symmetric spaces is the existence of an element $A\in\mathcal{H}$ such that $\mathcal{H}=\{B\in\mathcal{N}:[A,B]=0\}$. The matrix $\mathrm{ad}_A$ has only three distinct eigenvalues $0,\pm i$ and $[A,\mathcal{H}]=0,[A,X^\pm]=\pm iX^\pm$ for all $X^\pm\in\mathcal{P}^\pm$,

$\mathscr{P}=\mathscr{P}^{+}\oplus\mathscr{P}^{-}$. Let $e_{\pm\alpha}$ be a basis of the space $\mathscr{P}^{\pm}$. Then $[A,e_{\pm\alpha}]=\pm ie_{\pm\alpha}$, $[e_{\mu},[e_{\nu},e_{-\sigma}]] = R^{\alpha}_{\mu,\nu,-\sigma}\, e_{-\alpha}$, and $[e_{-\mu},[e_{-\nu},e_{\sigma}]]= R^{\alpha}_{\mu,\nu,-\sigma}\, e_{-\alpha}$, where $R^{\alpha}_{\mu,\nu,-\sigma}$ are the components of the curvature tensor defined at the initial point of the symmetric space[20] $N/H$.

For the matrices we choose from (23) the ansatz

$$T_t=\sum_{\alpha} r^{\alpha}(e_{\alpha}-e_{-\alpha})-i\sum_{\alpha,\beta}\Omega^{\alpha,\beta}[e_{\alpha},e_{-\beta}],\quad T_u=A,$$

$$T_y=0,\quad T_z=-i\sum_{\alpha} q^{\alpha}(e_{\alpha}+e_{-\alpha}), \tag{24}$$

where $r^{\alpha}$, $q^{\alpha}$, and $\Omega^{\alpha,\beta}$ are the arbitrary real-valued functions of $\rho$. Substituting (24) into Eqs. (23), we obtain

$$r^{\alpha}=\frac{d}{d\rho}q^{\alpha}+\frac{1}{\rho}q^{\alpha},\quad \Omega^{\alpha,\beta}=q^{\alpha}q^{\beta}-\Omega_0^{\alpha,\beta}+2\int\frac{d\rho}{\rho}q^{\alpha}q^{\beta}, \tag{25}$$

where $\Omega_0^{\alpha,\beta}=\Omega_0^{\beta,\alpha}=$ const and the functions $q^{\alpha}$ must satisfy the deformed equations of motion in quartic potentials

$$\frac{d^2}{d\rho^2}q^{\alpha}-\sum_{\mu,\nu,\sigma}R^{\alpha}_{\mu,\nu,-\sigma}q^{\mu}q^{\nu}q^{\sigma}+\sum_{\mu,\nu,\sigma}R^{\alpha}_{\mu,\nu,-\sigma}\Omega_0^{\nu,\sigma}q^{\mu}=-\frac{d}{d\rho}\left(\frac{1}{\rho}q^{\alpha}\right)$$
$$+2\sum_{\mu,\nu,\sigma}R^{\alpha}_{\mu,\nu,-\sigma}q^{\mu}\int\frac{d\rho}{\rho}q^{\nu}q^{\sigma}. \tag{26}$$

The corresponding linear system is obtained by substituting (24) and (25) into (22).

This work was supported by the Heisenberg–Landau Program and by the International Science Foundation (Grant NK8 000 and Grant MAL 000).

[1)]e-mail: ita@thsunl.jinr.dubna.su
[2)]e-mail: popov@thsunl.jinr.dubna.su

[1] V. E. Zakharov and A. B. Shabat, Funk. Anal. i Pril. **13**, 13 (1979); V. E. Zakharov and A. V. Mikhailov, Zh. Eksp. Teor. Fiz. **74**, 1953 (1978) [Sov. Phys. JETP **47**, 1017 (1978)].
[2] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, Berlin, 1987.
[3] V. A. Belinsky and V. E. Zakharov, Zh. Eksp. Teor. Fiz. **75**, 1952 (1978) [Sov. Phys. JETP **48**, 984 (1978)]; **77**, 3 (1979) [**50**, 1 (1979)].
[4] D. Maison, Phys. Rev. Lett. **41**, 521 (1978); J. Math. Phys. **20**, 871 (1979); Preprint MPI-PAE/PTh-3/82, 1982.
[5] A. V. Mikhailov and A. I. Yaremchuk, Nucl. Phys. B **202**, 508 (1982).
[6] S. P. Burtsev, V. E. Zakharov, and A. V. Mikhailov, Teor. Mat. Fiz. **70**, 323 (1987).
[7] L. Witten, Phys. Rev. D **19**, 718 (1979).
[8] R. S. Ward, Gen. Rel. Grav. **15**, 105 (1983); N. M. J. Woodhouse, Class. Quantum Grav. **4**, 799 (1987); **6**, 933 (1989); L. J. Mason and N. M. J. Woodhouse, Nonlinearity **1**, 73 (1988).
[9] A. A. Belavin, A. M. Polyakov, A. S. Schwarz, and Yu. S. Tyupkin, Phys. Lett. B **59**, 81 (1975).
[10] A. A. Belavin and V. E. Zakharov, Phys. Lett. B **73**, 53 (1978).
[11] R. S. Ward, Phys. Lett. A **61**, 81 (1977).
[12] R. S. Ward, Phil. Trans. R. Soc. Lond. A **315**, 451 (1985); Lect. Notes Phys. **280**, 106 (1987); Lond. Math. Soc. Lect. Notes Ser. **156**, 246 (1990).

[13] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering,* Cambridge University Press, Cambridge, 1991.
[14] T. A. Ivanova and A. D. Popov, Phys. Lett. A **170**, 293 (1992).
[15] F. Calogero and A. Degasperis, Commun. Math. Phys. **63**, 155 (1978).
[16] S. V. Manakov and V. E. Zakharov, Lett. Math. Phys. **5**, 247 (1981).
[17] M. Legaré and A. D. Popov, JETP Lett. **59**, 883 (1994).
[18] A. Lichnerowicz, *Géométrie des groupes de transformations, Dunod,* Paris, 1958; S. Kobayashi, *Transformation groups in differential geometry,* Springer-Verlag, Berlin, 1972
[19] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986; P. Winternitz, in *Partially Integrable Evolution Equations in Physics*, ed. by R. Conte and N. Boccara, NATO ASI Series C, Vol. 310, Kluwer Academic Publ., 1990, p. 515.
[20] A. Fordy, S. Wojciechowski, and I. Marshall, Phys. Lett. A **113**, 395 (1986).

Published in English in the original Russian journal. Reproduced here with stylistic changes by the Translation Editor.