Reduction of additional dimensions in nonuniform quantum Kaluza–Klein cosmological models

A. A. Kirillov

Scientific–Research Institute of Applied Mathematics and Cybernetics, 603005 Nizhniĭ Novgorod, Russia

(Submitted 13 June 1993)
Pis'ma Zh. Éksp. Teor. Fiz. **62**, No. 2, 81–86 (25 July 1995)

It is shown that the early stage of the evolution of the universe with $n<10$ is characterized by the fact that the volume of an arbitrary space-like hypersurface with dimension $m<n-2$ is compressed. It is shown that this behavior does not depend on the choice of the initial quantum state and can provide a mechanism for reducing the additional spatial dimensions. © *1995 American Institute of Physics.*

We know that different unification theories[1] predict the existence of additional spatial dimensions. It is assumed that at present, the size of the universe in the additional dimensions is of the order of Planck's length and the extra dimensions themselves are manifested in the form of ordinary matter, as a collection of scalar and vector fields. This picture, however, is no longer valid at the earliest times of the development of the universe (near the cosmological singularity), and it should be expected that during this period all dimensions play an equal role. This makes it possible to use different multidimensional theories of gravitation to describe the early universe and, at the same time, it raises the question of the mechanism of compactification of the additional dimensions.[2]

As one such mechanism it is possible to make use of the fact that near a singularity the local behavior of a nonuniform gravitational field is strongly anisotropic.[3–5] At the same time, an element of the spatial volume expands in certain directions and is compressed in other directions (Kasner regime). This behavior could be interpreted as dynamical reduction of the number of dimensions, except for one fact. First, for $n<10$ (n is the number of spatial dimensions) the Kasner regime is unstable — such regimes alternate — and, second, for $n\geqslant 10$ the number of directions in which compression occurs is found to depend on the position in space. In this sense, the dimension of the space should depend on our location in the universe.

In the present paper we study the quantum behavior of nonuniform multidimensional models near a singularity and we show that, at least for $n<10$, the dimension 3 is identifiable in the sense that in the process of cosmological expansion the space is compressed along arbitrary hypersurfaces $m<n-2$.

As indicated in Ref. 5 , the behavior of a nonuniform gravitational field near a singularity can be described on the basis of an asymptotic model. The $(n+1)$-dimensional interval for such a model is represented in the Kasner form

$$ds^2=-N^2dt^2+\sum_{a=0}^{n-1} e^{q^a}(l^a)^2, \qquad (1)$$

where $l^a = l^a_\alpha\, dx^\alpha$ are Kasner vectors which contain only $n(n-1)$ arbitrary functions of the spatial coordinates. The dynamics of the gravitational field is determined in leading order by the behavior of only the scale functions q^a, while the vectors l^a_α play a passive role. The evolution of the scale functions is described by the action[5]

$$I=\int_S\left\{p_a\frac{\partial q^a}{\partial t}-\lambda\left\{\sum p^2-\frac{1}{n-1}\left(\sum p\right)^2+U\right\}\right\}d^nxdt, \tag{2}$$

where λ is expressed in terms of the lapse function $\lambda = N/\sqrt{g}$. The potential in Eq. (2) can be represented in the form $U=-gR^n=\Sigma\lambda_A g^{\sigma_A}$, where the coefficients λ_A are functions of all dynamical variables and their derivatives, and the exponents σ_A are given by the expressions $\sigma_{abc}=1+Q_a-Q_b-Q_c$ $(b\neq c)$, where $Q_a=q^a/\Sigma q$ are the anisotropy parameters. The asymptotic behavior of the potential U in the limit $g=\exp(\Sigma q)\to 0$ is modeled by potential walls,[5,6]

$$g^{\sigma_A}\to\theta_\infty[\sigma_A(Q)]=\begin{cases}+\infty, & \sigma_A<0\\ 0, & \sigma_A>0.\end{cases}$$

This potential is found to be independent of the Kasner vectors.

The configuration space M of the system (2) (superspace) is represented as a direct product $M=\Pi_{x\in S}M_x$. Since we are interested in the behavior of the local characteristics of the space at a single point $x\in S$, it is sufficient to study only one term M_x in this product. This limitation is possible on account of the fact that because of the large-scale nature of the gravitational field, in the asymptotic limit $g\to 0$ the dynamics of the M_x degrees of freedom do not depend on the other terms in the direct product.[5]

The space M_x is an ordinary n-dimensional pseudo-Euclidean space. In the harmonic variables the part of the action (2) that refers to M_x assumes a form that is formally identical to the action for a relativistic particle [for simplicity we set $(\Delta x)^n=1$]:

$$I=\int\left\{P_a\frac{\partial z^a}{\partial t}-\lambda'(P_i^2+U-P_0^2)\right\}dt, \tag{3}$$

where $\lambda'=\lambda/n(n-1)$, and the variables z^a are determined by the relation $q^a=A^a_j z^j+z^0$ $(j=1,\ldots,n-1)$ with a constant matrix (see Ref. 5)

$$A^a_j=\sqrt{\frac{n(n-1)}{j(j+1)}}(\theta^a_j-j\delta^a_j),\quad \theta^a_j=\begin{cases}1, & j>a,\\ 0, & j\leqslant a.\end{cases}$$

As shown in Ref. 7 , quantization of such a system is performed just as for relativistic particles.[8] The existence of a Hamiltonian coupling leads to the Wheeler–de Witt equation[9]

$$(-\Delta+U+\xi P)\Psi=0, \tag{4}$$

where Ψ is the wave function describing the quantum states of the degrees of freedom M_x,

$$\Delta=\frac{1}{\sqrt{-G}}\partial_A\sqrt{-G}G^{AB}\partial_B .$$

G_{AB} is a metric given by the interval

$$\delta\Gamma^2=\frac{1}{4\lambda'}((\delta z^i)^2-(\delta z^0)^2),$$

and P is the scalar of the curvature on M_x. The value of the constant ξ must be chosen as $\xi=(n-2)/4(n-1)$, which renders Eq. (4) conformally invariant, which reflects that the lapse function λ was chosen arbitrarily.

Using the Misner–Chitra variables[6,5] (see also Ref. 10) ($\mathbf{y}=y^j$)

$$z^0=-e^{-\tau}\frac{1+y^2}{1-y^2},\quad \mathbf{z}=-2e^{-\tau}\frac{\mathbf{y}}{1-y^2},\quad y=|\mathbf{y}|\leqslant 1, \tag{5}$$

we can represent the metric on M_x in the form

$$\delta\Gamma^2=\frac{e^{-2\tau}}{4\lambda'}\left(\frac{4(\delta y^j)^2}{(1-y^2)^2}-(\delta\tau)^2\right). \tag{6}$$

In these variables the anisotropy parameters Q_a and therefore the potential $U(Q)$ do not depend on the time variable τ:

$$Q_a(y)=\frac{1}{n}\left\{1+\frac{2A^a_j y^j}{1+y^2}\right\}.$$

For simplicity, we employ below the gauge $4\lambda' e^{2\tau}=1$.

The space-like part of the configuration space M_x is a $(n-1)$-dimensional Lobachevskiĭ space, and the potential U limits its part K (Ref. 5)

$$\sigma_{abc}=1+Q_a-Q_b-Q_c\geqslant 0,\quad a\neq b\neq c. \tag{7}$$

Then the complete orthonormal set $\{u_p, u_p^*\}$ of solutions of Eq. (4) will then consist of functions of the form

$$u_J=\frac{1}{\sqrt{2k_J}}\exp(-ik_J\tau)\varphi_J(y), \tag{8}$$

where φ_J are the eigenfunctions of the Laplace–Beltrami operator

$$\left(\Delta_y+k_J^2+\frac{(n-2)^2}{4}\right)\varphi_J(z)=0,\qquad \varphi_J|_{\partial K}=0, \tag{9}$$

and the operator Δ_y is constructed using the metric $dl^2=h_{ij}\,dy^i dy^j=4(dy)^2/(1-y^2)^2$. In the case $n<10$ the region K has a finite volume and J assumes only discrete values ($J=0,1,2\ldots$). For $n\geqslant 10$ the volume of the region K is infinite and the spectrum is continuous.

The probabilistic interpretation is introduced by separating the positive-frequency spectrum H^+ in the space H of the solutions of Eq. (4) (see Refs. 7 and 8). Therefore, an arbitrary wave function Ψ, which describes the physical state of the gravitational field at the point x, can be represented in the form

$$\Psi=\sum_J A_J u_J, \quad \langle\Psi|\Psi\rangle=\sum |A_J|^2=1, \tag{10}$$

where A_J are arbitrary constants, determined so as to satisfy the initial conditions. We note that the states (8) play the role of stationary states of the gravitational field, but the geometry corresponding to these states is nonstationary (since the metric contains the time variable τ explicitly).[7] The probability that the scale functions are localized at the point $O=(y^i,\tau) \in M_x$ is given by the expression $P(y,\tau)=|\langle\Phi(y,\tau)|\Psi\rangle|^2$, where $\Phi(y,\tau)=\Sigma_J\sqrt{k_J}u_J^*(y,\tau)u_J$ are localized Newton–Wigner states.[8] Therefore, for an arbitrarily chosen initial state, we obtain $P(y,\tau)=|\Sigma_J\sqrt{k_J}u_J^*(y,\tau)A_J|^2$.

We now consider an arbitrary m-dimensional hypersurface $\Xi^m\subset S$. An element of volume of this hypersurface has the form $dV^m=\Sigma g^{\mu_{a_1,\dots,a_m}}C_{a_1,\dots,a_m}l^{a_1}\wedge\dots\wedge l^{a_m}$, where $\mu_{a_1,\dots,a_m}=\frac{1}{2}\Sigma_{i=1}^m Q_{a_i}$ and $C_{a_1,\dots,a_m}(x)$ are arbitrary constant functions which give the position of the hypersurface in S. Therefore, the behavior of this element as a function of time is determined by quantities of the type g^{μ_m}. In the quantum theory quantities such as the volume are operators and must be averaged over the quantum state.

It turns out that in the asymptotic limit ($g\to 0$) for $n\leqslant 9$ the behavior of different m-dimensional volumes as a function of time exhibits universal properties. This is because the main contribution to averages of the type $\langle g^{\mu_m}\rangle$ comes from a small neighborhood of points $y^*\in K$, where the exponents μ_m assume minimum values. Such points lie on the boundary ∂K, and the minimum values of the exponents are given by the expressions with $\mu_m^*=-m(n-m-2)/2(n+m)$ for $m<n-2$, $\mu_{n-2}^*=\mu_{n-1}^*=0$, and $\mu_n\equiv 1/2$. Specifically, $2\mu_1$ determines the minimum admissable value $Q_{\min}$ of the anisotropy parameters.[5] Since $\varphi_J(\partial K)=0$, near the boundary ∂K we have $\varphi_J\approx\eta_J(\mu-\mu^*)$ and the probability density assumes the form (we assume that $n>3$):

$$P_\tau(\mu)=\int_K P(y,\tau)\delta(\mu-\mu(y))\sqrt{h}d^{m-1}y\approx B_m(\tau)(\mu-\mu^*)^n, \quad \mu\to\mu^*.$$

In the limit $g\to 0$ we thus obtain for the moments of the functions g^{μ_m} the expression ($L>0$)

$$\langle(g^{\mu_m})^L\rangle=D_m(L,\tau)\frac{(g_*^{\mu_m^*})^L}{(L\ \ln 1/g_*)^{n+1}}, \tag{11}$$

where $g_*=g(\tau,y^*)$, and D_m is a slowly (logarithmically) varying function of time, which depends on the choice of the initial quantum state. For $m<n-2$ we have $\mu_m^*<0$ and therefore the volume of an arbitrary hypersurface Ξ^m, whose dimension is less than $n-2$, is found to be compressed (we note that this still does not solve the problem of compactification of the extra dimensions, and only shows the initial tendency toward such compactification; the question of the later stage and its stability remains open).[2] In the early universe the number of spatial dimensions can thus be effectively reduced to three. The law of expansion of the remaining three-dimensional space [complementary to the $(n-3)$-dimensional hypersurface] can be estimated as $V_3 \sim g^{(3/2)k}$, where $k=\frac{2}{3}(\frac{1}{2}-\mu_{n-3})=(n-2)/(2n-3)$, which corresponds to an effective equation of state of the matter $p=(n/3n-6)\epsilon$ (we recall that near a singularity we have $g \sim t^2$, where t is the synchronous time).

The situation is fundamentally different for the dimensions exceeding $n=9$. In this case the potential in Eq. (4) no longer limits the space-like part of the configuration space (the region K has an infinite volume), and therefore the stationary states (8) are no longer localized in terms of the exponents μ. If a state (wave packet) which is localized with respect to μ is prepared, then such a packet starts to spread with time and its center of gravity recedes to infinity in the configuration space (in the classical theory this corresponds to the fact that the evolution of the metric is described by a stable Kasner regime). Different averages (and the number of collapsing directions) are therefore found to depend strongly on the choice of the initial state.

In conclusion, we note that in the classical theory with $n<10$ the evolution of the metric becomes stochastic,[5] and the properties of the stochastic process are described by an invariant measure (see also Ref. 10). Using this measure, it is possible to estimate the behavior of the average m-dimensional volumes. It turns out that these estimates are identical to expression (11) up to a logarithmic factor [with the substitution $n \rightarrow n-2$ in Eq. (11) and D_m which are now constants]. We note, however, that in the classical theory such estimates are of limited value, since the need for a probabilistic description arises as a result of the increase in the uncertainty in prescribing the initial conditions. In the quantum theory, however, the description is probabilistic from the onset and only the average values of different operators have physical (experimental) status.

I wish to thank G. Montani, who called my attention to the fact that the Kasner behavior of the metric in the process of cosmological collapse with exponents of different sign can be interpreted as an effective dynamical reduction of the space dimension.

This work is partially supported by the Russian Research Project "Kosmomikrofizika."

[1]M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Cambridge University Press, N.Y., 1988.
[2]A. Chodos and S. Detweiler, Phys. Rev. D **21**, 2176 (1980); M. Demianski, Z. Golda, M. Heller, and M. Szydlowski, Class. Quantum Grav. **3**, 1196 (1986); Class. Quantum Grav. **5**, 733 (1988); K. Maeda and P. Y. Pang, Phys. Lett. B **180**, 29 (1986).
[3]V. A. Belinskiĭ, E. M. Lifshitz, and I. M. Khalatnikov, Zh. Éksp. Teor. Fiz. **62**, 1606 (1972) [Sov. Phys. JETP **35**, 838 (1972)]; V. A. Belinskiĭ and I. M. Khalatnikov, Zh. Éksp. Teor. Fiz. **63**, 1121 (1972) [Sov. Phys. JETP **36**, 591 (1973)].
[4]J. Demaret, M. Henneaux, and P. Spindel, Phys. Lett. B **164**, 27 (1985); J. Demaret *et al.*, Phys. Lett. B **175**, 129 (1986); A. Hosoya, L. G. Jensen, and A. Stein-Schabes, Nucl. Phys. B **238**, 657 (1987); Y. Elskens and M. Henneaux, Nucl. Phys. B **290**, 111 (1987).
[5]A. A. Kirillov, Zh. Éksp. Teor. Fiz. **103**, 721 (1993) [JETP **76**, 355 (1993)]; A. A. Kirillov and V. N. Melnikov, Phys. Dev. D **52**, 723 (1995); G. Montani in *Proceedings of the 7th Marcel Grossmann Meeting on General Relativity*, WSP, Singapore, 1994.
[6]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* **2**, Freeman, San Francisco, 1973.
[7]A. A. Kirillov, Int. J. Mod. Phys. D **3**, 431 (1994); JETP Lett. **55**, 561 (1992); A. A. Kirillov and V. N. Melnikov, Preprint CBPF-NF-037/95, Rio de Janeiro, Brazil, 1995.

[8]S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Harper and Row, New York, 1961.
[9]B. S. de Witt, Phys. Rev. **160**, 1113 (1967).
[10]V. D. Ivashchuk, A. A. Kirillov, and V. N. Mel'nikov, JETP Lett. **60**, 235 (1994).

Translated by M. E. Alferieff