Instantons and monopoles in maximal abelian projection of *SU*(2) gluodynamics

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We show that the instantons induce the Abelian monopoles in the Maximal Abelian Projection of $SU(2)$ gluodynamics. As an example we consider the case of one instanton and the case of a set of instantons arranged along a straight line. The Abelian monopoles which are induced by instantons may play some role in the confinement scenario. © *1995 American Institute of Physics.*

1. INTRODUCTION

The confinement phenomenon in QCD is one of the most important problems in the non-Abelian theories. Although various instanton models of QCD yield vanishing string tension for the QCD string,[1] the instantons seem to play an important role. For instance the correlation functions of various colorless quark bound states are well described by the instanton contributions.[2]

An explanation of the confinement phenomenon was proposed by 't Hooft[3] and Mandelstam,[4] who conjectured that the infrared properties of the confining QCD vacuum are similar to those of the (dual) superconductor. The most convenient way to think about this analogy is to partially fix the $SU(N)$ gauge degrees of freedom, leaving the $[U(1)]^{N-1}$ group unfixed.[5] Such a partial gauge is usually called an Abelian projection. Under the Abelian transformations, the diagonal elements of the gluon field transform as gauge fields, and, due to the compactness of the $U(1)$ gauge group, Abelian monopoles exist. If they are condensed, the string between the colored charges is formed as the dual analogue of the Abrikosov string in a superconductor, the monopoles playing the role of the Cooper pairs.[3,4]

Many numerical simulations have demonstrated the dual superconductor mechanism in $SU(2)$ lattice gauge theory (for a review see Ref. 6); most of the results have been obtained in the so-called Maximal Abelian (MA) projection.[7] However, there is also the Abelian projection, in which the role of monopoles is played by the other topological objects—by "minipoles."[8]

Below we show that several instantons in the 't Hooft ansatz, with the centers placed on a straight line, lead to an Abelian monopole current along this line. For the sake of simplicity we discuss only the $SU(2)$ gauge group.

2. FROM INSTANTON TO ABELIAN MONOPOLE

The $SU(2)$ instanton field configuration in the 't Hooft ansatz is given by the equation

$$A^a_\mu=\frac{1}{g}\bar{\eta}^a_{\mu\nu}\partial_\nu f(x), \tag{1}$$

where $\bar{\eta}^a_{\mu\nu}$ is the 't Hooft symbol and g is the coupling constant. The function $f(x)$ is given by

$$f(x)=f^I(x)=\ln\left[1+\frac{\rho^2}{t^2+r^2}\right], \tag{2}$$

where ρ is the size of the instanton, t is the time, and r is the spatial radius.

The gauge conditions which define the MA projection are given by the formula:[7]

$$(\partial_\mu \pm igA^3_\mu)A^\pm_\mu=0, \tag{3}$$

where $A^\pm_\mu=A^1_\mu \pm iA^2_\mu$, the $SU(2)$ generators are $t^a=\sigma^a/2$, and σ^a are the Pauli matrices. Let us rotate the instanton field (1), (2) by the $SU(2)$ matrix Ω:

$$\Omega=\begin{pmatrix} \cos\phi e^{i\theta} & \sin\phi e^{i\chi} \\ -\sin\phi e^{-i\chi} & \cos\phi e^{-i\theta}\end{pmatrix}, \tag{4}$$

where

$$\chi=\Delta-\alpha/2, \quad \theta=-\Delta-\alpha/2, \quad \phi=\gamma/2. \tag{5}$$

Here α and γ are azimuthal and polar angles of the coordinate system in the given time slice, and $\Delta(x)$ is an arbitrary function. As can be easily checked, the instanton field A which is rotated by this matrix Ω,

$$A_\mu \to A^{(\Omega)}_\mu=\Omega^+A_\mu\Omega-\frac{i}{g}\Omega+\Omega+\partial_\mu\Omega, \tag{6}$$

satisfies the MA projection conditions (3).

The field strength tensor

$$G_{\mu\nu}[A]=\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu,A_\nu] \tag{7}$$

with the transformed field $A^{(\Omega)}_\mu$ satisfies the equation

$$\epsilon_{\mu\nu\alpha\beta}\partial_\nu G_{\alpha\beta}[A^{(\Omega)}]=\frac{2\pi}{g}j_\mu\sigma^3, \tag{8}$$

where j_μ is given by the following equation:

$$j_\mu(x)=\delta^{(3)}(\mathbf{r})\delta_{\mu 0}. \tag{9}$$

Equation (8) is none other than the definition of the monopole singularities in the gauge field A^3_μ, since the only third component $G^3[A^{(\Omega)}]$ contributes[1)] to the right-hand side of Eq. (8). The monopole current $j_\mu(x)$ in Eq. (9) is the straight line along the time direction which crosses the center of the instanton (which is the origin of our coordinate system).

Note that the monopole trajectory does not depend on the choice of the function $\Delta(x)$ used in the definition of the matrix Ω [Eqs. (4) and (5)]. It can be shown that this function corresponds to the residual $U(1)$ degrees of freedom we leave unfixed. $\Delta(x)$ does not affect $U(1)$-invariant quantities, while $U(1)$-variant quantities (e.g., the position of the Dirac string which is attached to the Abelian monopole) do change if the function $\Delta(x)$ is changed.

The choice (5) for the $SU(2)$-angles θ, χ, and ϕ is not unique. Let us choose an arbitrary unit vector n_μ and define three vectors $m_\mu^{(a)}$, $a=1,2,3$ as follows:

$$m_\mu^{(k)} = \bar{\eta}_{\mu\nu}^{a} n_\nu . \tag{10}$$

Using the properties of the 't Hooft symbols $\bar{\eta}_{\mu\nu}^{a}$, we can easily show that vectors n and $m^{(a)}$ form the basis in 4D Euclidean space-time:

$$m_\mu^{(k)} m_\mu^{(l)} = \delta^{kl}; \quad m_\mu^{(k)} n_\mu = 0 \quad k=1,2,3. \tag{11}$$

Let us choose the $SU(2)$-angles θ, χ, and ϕ as in Eq. (5) but with the azimuthal, (α) and polar (γ) angles defined with respect to the new 3D basis $\{m_\mu^{(a)}\}$. The matrix Ω [Eq. (4)], with the new angles θ, χ, and ϕ, will be referred to as $\Omega[n]$. It can be easily verified that the transformation of the instanton field (1), (2) by the matrix $\Omega[n]$ satisfies the MA projection conditions.[3]

Repeating all the above calculations but with the new matrix Ω, we get for the Abelian monopole current which is to be determined from Eq. (8):

$$j_\mu = \delta^{(3)}(y) n_\mu , \tag{12}$$

where $y^k = m_\nu^k x_\nu$. The monopole trajectory is the straight line which crosses the center of the instanton and has the direction n_μ.

3. TWO AND MANY INSTANTONS

The N-instanton field configuration $A^{N \cdot I}$ may be written in the 't Hooft ansatz (1), where the function $f(x)$ is given by the formula:

$$f(x) = f^{N \cdot I}(x) = \ln\left[1 + \sum_{i=1}^{N} \frac{\rho_i^2}{(x-x_i)^2} \right]. \tag{13}$$

Here x_i is the position of the center of ith instanton, and all the instantons have the same color orientations.

Consider first the case of two instantons, $N=2$. Let us rotate the field (1), (13) by the matrix $\Omega[n]$, where the vector n_μ is oriented along the line which connects the centers x_1 and x_2 of these instantons. A direct evaluation shows that the rotated field satisfies the MA projection conditions (3). Substituting the field $A=A^{N \cdot I}$ and the matrix $\Omega=\Omega[n]$ into Eq. (8), we finally obtain expression (12) for the Abelian monopole current. Note that the sizes ρ_i of the instantons are not important.

Thus we can conclude that any two instantons in the 't Hooft ansatz carry in the MA projection a straight-line Abelian monopole current j_μ which goes through the centers of these instantons. Note that we did not prove the uniqueness of the solution of equation (3) with respect to the gauge transformations (6): the existence of other monopoles trajectories is not ruled out.

The matrix $\Omega[n]$ described above rotates an arbitrary configuration of instantons within the MA projection, provided that the centers of all these instantons are on the same straight line and the color orientations are also the same. As in the two-instanton case, the sizes of the instantons are irrelevant and the resulting Abelian monopole trajectory passes through the centers of instantons.

The particular instanton configuration considered here corresponds to the BPS-monopole:[9] the distance between any two neighbor centers should be fixed, and the sizes of all the instantons should be equal to infinity. The BPS-monopole trajectory (which is a straight line) coincides with the trajectory of the Abelian monopole. This case was already discussed in Ref. 10.

There are more general field configurations which give, in the MA projection, Abelian monopole trajectories which are arbitrary straight lines with the direction n_μ. These field configurations may be represented in the 't Hooft ansatz (1) with the function $f(x)$ being dependent only on the combinations $t' = nx$ and $r' = \sqrt{x^2 - (nx)^2}$. The resulting monopole lines pass through the origin of our coordinate system.

4. CONCLUSIONS

Our calculations show that the single instanton field configuration can be rotated within the MA projection by various gauge transformations, each of which is parametrized by an arbitrary vector n_μ. The induced monopole currents are straight lines with the direction n_μ. Many-instanton configurations induce Abelian monopoles as well.

Calculations in lattice gluodynamics show that the Abelian monopoles are responsible for the confinement in the MA projection.[6] The contribution of the Abelian monopoles to the Wilson loop leads to the area law with the correct string tension coefficient.[11,12] Moreover, it has been observed[13] that the long monopole loops alone are responsible for the behavior of the string tension in the confinement phase of $SU(2)$ QCD. We found that monopole trajectories of this type may be induced by instantons (and BPS monopoles) into the MA-projected $SU(2)$ vacuum. Therefore, the monopoles induced by instantons may in principle be important for the confinement scenario.

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[i]Note that the field strength tensor (7) transforms under singular gauge transformations Ω as follows: $G_{\mu\nu} \to G_{\mu\nu}[A^{(\Omega)}] = \Omega^+ G_{\mu\nu}[A]\Omega + G^{\mathrm{sing}}_{\mu\nu}[\Omega]$, where $G^{\mathrm{sing}}_{\mu\nu}[\Omega] = -i\Omega^+(x)[\partial_\mu\partial_\nu - \partial_\nu\partial_\mu]\Omega(x)$. The matrix (4) is singular and therefore $G^{\mathrm{sing}}_{\mu\nu}[\Omega]$ is not zero. It can be also shown that only $G^{\mathrm{sing}}_{\mu\nu}[\Omega]$ contributes to the right-hand side of Eq. (8).

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