Spontaneous breaking of chiral invariance by an external magnetic field in (2+1) dimensions

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The phase structure of the three-dimensional Gross–Neveu model in an external magnetic field H in the presence of a chemical potential μ is investigated. The critical curve $\mu_c(H)$ which separates the zero-mass symmetric theory from the massive theory with spontaneous breaking of the chiral invariance of the phase is constructed in the (μ,H) plane. The behavior of the critical curve in the limits $H\rightarrow\infty$ and $H\rightarrow 0$ is determined.

It was recently shown[1] that an external magnetic field H is a catalyst for spontaneous breaking of the chiral (flavor) symmetry in (2+1)-dimensional quantum field theories. In Ref. 1 it was shown that for $H\neq 0$ the condensate of the spinor fields is different from zero in the zero-mass quantum electrodynamics (QED_3) and also in the three-dimensional model with a four-fermion interaction. The effect was investigated in greater detail in subsequent works.[2,3] Specifically, it was shown[3] that within QED_3 the formation of the condensate $\langle\bar{\psi}\psi\rangle$ is unstable relative to the temperature T and the chemical potential μ; i.e., for $T\neq 0$ and $\mu\neq 0$ the spinor-field condensate is equal to zero.

In view of the discovery of the "new" effect, we wish to say, first of all, that even in Ref. 4 the three-dimensional Gross–Neveu (GN) model was studied with $H\neq 0$ and $T\neq 0$. It was shown there that an external magnetic field gives rise to spontaneous breaking of the chiral symmetry. Moreover, the critical temperature $T_c(H)$ (in contrast to QED_3 it is not zero), below which the fermion masses are different from zero, was found (i.e., $\langle\bar{\psi}\psi\rangle\neq 0$, and the symmetry is broken spontaneously), while for $T>T_c(H)$ the system passes into a zero-mass phase via a second-order phase transition.

In the present paper we investigate the combined effect of an external magnetic field and a chemical potential on the phase structure of the three-dimensional GN model. The Lagrangian of the model is

$$L=\bar{\psi}_k i\hat{\partial}\psi_k+\frac{G}{2N}(\bar{\psi}_k\psi_k)^2, \tag{1}$$

where the summation over k from one to N is implied, and ψ_k is a four-component Dirac spinor for each fixed k. In this formalism

$$\gamma^0=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix},\quad \gamma^1=\begin{pmatrix}i\sigma_1 & 0\\ 0 & -i\sigma_1\end{pmatrix},\quad \gamma^2=\begin{pmatrix}i\sigma_2 & 0\\ 0 & -i\sigma_2\end{pmatrix},$$

and the Lagrangian (1) is invariant under a discrete chiral transformation

$$\psi_k\rightarrow\gamma^5\psi_k,\quad \gamma^5=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \tag{2}$$

where σ_i are Pauli matrices, and I is a 2×2 unit matrix. We believe that the results of the present paper could be helpful in explaining different planar effects, including the quantum Hall effect and high-T_c superconductivity, for the description of which the external magnetic field H and chemical potential μ must be invoked.

We consider first the case $H,\ \mu=0$. Introducing an auxiliary scalar field $\sigma\sim\bar{\psi}_k\psi_k$, we can write the effective potential of the model (1) in the leading order of an expansion in $1/N$ as[4,5]

$$\frac{1}{N}V_0(\sigma)=\frac{\sigma^2}{2G}-2\int\frac{d^3p}{(2\pi)^3}\ln(p^2+\sigma^2). \tag{3}$$

The integration in Eq. (3) extends over the region $0\leqslant p^2\leqslant\Lambda^2$ in Euclidean momentum space. The procedure for eliminating the ultraviolet divergences from expressions of the type (3) is described in Ref. 5, so that we immediately present the renormalized effective potential

$$\frac{1}{N}V_0(\sigma)=\frac{\sigma^2}{2f}+\frac{|\sigma|^3}{3\pi}, \tag{4}$$

$$\frac{1}{f}\equiv\frac{1}{G}-\frac{2\Lambda}{\pi^2}=\frac{1}{G(m)}-\frac{2m}{\pi}, \tag{5}$$

where $G(m)$ is the renormalized coupling constant, and m is the normalization point.

It follows from Eq. (5) that the constant f does not depend on either the normalization point m or the parameter Λ; i.e., expression (4) is finite and renormalization-invariant. It follows from Eq. (4) that for $f>0$ the absolute minimum of the potential lies at the point $\sigma=0$, so that the chiral invariance of the theory is not broken. For $f<0$ the point of the global minimum of the potential is $\sigma_0=-\pi/f$. In this case the mass $M\equiv\sigma_0$ arises spontaneously for fermions and the symmetry of the model under the transformation (2) is broken spontaneously. [If the constant $g\equiv\Lambda G$ is used as an independent parameter,[1] then we find for $g<g_c=\pi^2/2$ a zero-mass symmetric phase of the theory. In this case, as follows from Eq. (5), $f>0$. Let $g>g_c$ (i.e., $f<0$). Then a massive phase with spontaneous breaking of the chiral invariance is realized in the system.]

We now consider the case $H,\mu\neq 0$. Omitting the quite tedious calculations, we present immediately an expression for the effective potential:

$$V_{H\mu}(\sigma)=V_H(\sigma)-\frac{NeH}{2\pi}\sum_{n=0}^{\infty}\alpha_n\Theta(\mu-\epsilon_n)(\mu-\epsilon_n). \tag{6}$$

[Equation (6) is derived similarly to the derivation of the effective action in QED_3 in the one-loop approximation with $H,\mu,T\neq 0^6$ (Ref. 6).] Here $\alpha_n=2-\delta_{n0}$, $\Theta(x)$ is the Heaviside function, $\epsilon_n=(\sigma^2+2eHn)^{1/2}$, e is the electric charge, and $V_H(\sigma)$ is the effective potential with $H\neq 0$, $\mu=0$: (Refs. 4 and 7):

$$\frac{1}{N}V_H(\sigma)=\frac{1}{N}V_0(\sigma)+\frac{eH}{4\pi^{3/2}}\int_0^\infty\frac{dx}{x^{3/2}}\,e^{-x\sigma^2}\left[\coth(eHx)-\frac{1}{eHx}\right]$$
$$=\frac{\sigma^2}{2f}+\frac{eH|\sigma|}{2\pi}-\frac{(2eH)^{3/2}}{2\pi}\zeta\left(-\frac{1}{2},\frac{\sigma^2}{2eH}\right), \tag{7}$$

where $\zeta(s,v)$ is the generalized Riemann zeta function.[8] The point $\sigma_0(H)$ of the global minimum of the function (7) has the following properties.[4] As $H\to 0$,

$$\sigma_0(H)=\begin{cases} feH/2\pi+..., & f>0\\ M[1+(eH)^2/12M^4+...], & f<0\end{cases}. \tag{8}$$

Here M is the mass of the fermions at $H=0$. As $H\to\infty$, we have, irrespective of the sign of the constant f

$$\sigma_0(H)\simeq\sqrt{0.2eH}. \tag{9}$$

Let us analyze the behavior of the point of the global minimum of the potential $V_{H\mu}(\sigma)$ as a function of μ and H. The equation expressing the steady state of the function (6) on the interval $\sigma\geqslant 0$ has the form

$$0=\frac{\sigma}{f}+\frac{eH}{2\pi}-\frac{\sigma\sqrt{2eH}}{2\pi}\zeta\left(\frac{1}{2},\frac{\sigma^2}{2eH}\right)+\frac{eH}{2\pi}\sum_{n=0}^{\infty}\alpha_n\Theta(\mu-\epsilon_n)\frac{\sigma}{\epsilon_n}. \tag{10}$$

Let us divide the parameter plane (μ,H) for $\mu,H\geqslant 0$ into the regions Ω_n:

$$(\mu,H)=\bigcup_{n=0}^{\infty}\Omega_n;\quad \Omega_n=\{(\mu,H):2eHn\leqslant\mu^2\leqslant 2eH(n+1)\}. \tag{11}$$

It is obvious that in the region Ω_0 only the first term under the summation sign in Eq. (10) is different from zero, in Ω_1 the first and second terms are different from zero, and so on. In what follows, we shall require the following very important expansion of the ζ function[8]

$$\zeta(\tfrac{1}{2},\vartheta)=\sum_{i=0}^{k}(\vartheta+i)^{-1/2}-2\sqrt{k+\vartheta}-\sum_{i=k}^{\infty}f_i(\vartheta), \tag{12}$$

where

$$f_k(\vartheta)=\frac{1}{2}\int_k^{k+1}\frac{(u-k)du}{(u+\vartheta)^{3/2}}>0. \tag{13}$$

Let $f>0$. We construct in the (μ,H) plane the curve $\mu=\sigma_0(H)$, where $\sigma_0(H)$ is the point of the global minimum of the potential (7). It is evident from Eqs. (8) and (9) that for sufficiently high and low values of the field H the curve $\mu=\sigma_0(H)$ passes through the region Ω_0. Let $(\mu,H)\in\Omega_0$, where the steady-state equation (10) has the form [here we employed the expansion (12) with $k=0$]

$$\frac{\sigma}{f}+\frac{\sigma^2}{\pi}+\frac{\sigma\sqrt{2eH}}{2\pi}\sum_{i=0}^{\infty} f_i\left(\frac{\sigma^2}{2eH}\right)-\frac{eH}{2\pi}[1-\Theta(\mu-\sigma)]=0. \tag{14}$$

It is obvious that for $\mu>\sigma_0(H)$ Eq. (14) has only one solution: $\sigma_1=0$. If $\mu<\sigma_0(H)$, then one other solution, in addition to σ_1, will appear for Eq. (14): $\sigma_2=\sigma_0(H)$. Similar calculations can be performed for any region Ω_n. Therefore, for $f>0$ the points of the plane (μ,H) which lie above the curve $\mu=\sigma_0(H)$ correspond to the effective potential (6) which has a global minimum at $\sigma_1=0$. For $\mu>\sigma_0(H)$ we thus have a zero-mass, chirally invariant phase of the theory. If $\mu<\sigma_0(H)$, then Eq. (10) will have two solutions: $\sigma_1=0$ and $\sigma_2=\sigma_0(H)$. The equation

$$V_{H\mu}(0)=V_{H\mu}(\sigma_0(H)) \tag{15}$$

therefore determines the critical curve $\mu=\mu_c(H)$ which separates in the (μ,H) plane the zero-mass phase [for $\mu>\mu_c(H)$] from the massive phase [for $\mu<\mu_c(H)$] in which the chiral symmetry of the theory is broken spotaneously. A first-order phase transition occurs when the critical curve is crossed, since the mass of the fermions in the case changes abruptly from zero to $\sigma_0(H)$. Since $\mu_c(H)\leqslant\sigma_0(H)$, for moderately low and moderately high values of H the critical curve lies in the region Ω_0, where Eq. (15) has the form

$$\mu_c(H)=\frac{2\pi}{eH}[V_H(0)-V_H(\sigma_0(H))]. \tag{16}$$

For small values of H it follows from Eqs. (8) and (16) that

$$\mu_c(H)\cong\frac{2\pi}{eH}\frac{dV_H(0)}{d\sigma}\sigma_0(H)=\sigma_0(H). \tag{17}$$

In the limit $H\to\infty$ we obtain from Eqs. (9) and (16)

$$\mu_c(H)\sim\sqrt{eH}. \tag{18}$$

If $f<0$, then the analysis of the potential (6) becomes much more complicated. The problem is that in this case the equation for the steady state (10) can have solutions other than $\sigma_1=0$ and $\sigma_2=\sigma_0(H)$. As a result, we can obtain exact solutions only for sufficiently high values of the magnetic field. Omitting the detailed calculations, we note that for $M=-\pi/f$

$$2M^2<eH \tag{19}$$

the potential $V_{H\mu}(\sigma)$ can have, just as in the case $f>0$, not more than two stationary points $\sigma_{1,2}=0$ and $\sigma_0(H)$. In the region (19) of the (μ,H) plane the critical curve $\mu_c(H)$ is therefore determined by Eq. (15). For sufficiently high values of H, when $\mu_c(H)$ lies in the region Ω_0, the critical curve has the form (16) and in the limit $H\to\infty$ the quantity $\mu_c(H)$ behaves as $\sqrt{eH}$.

In summary, we have shown that in the three-dimensional Gross–Neveu model the chiral invariance, which is broken spontaneously by an external magnetic field, is necessarily restored for some value of the chemical potential $\mu_c(H)\neq 0$. As $H\to\infty$, the critical curve $\mu_c(H)\sim\sqrt{eH}$, and for low values of H and $f>0$ $\mu_c(H)$ is equal to the dynamic fermion mass $\sigma_0(H)$.

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