Characteristic features of some typical spontaneous intensity collapse processes in unstable media

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The results of a local analysis of spontaneous intensity collapse processes in unstable media in the nonlinear geometric optics approximation for situations of the "general position" are presented. It is shown that these processes correspond to fold and cusp singularities which are typical for smooth mappings. It is shown that when the effect of weak dispersion is taken into account, a pulse is described near points of self-peaking by a special nonlinear function — an isomonodromic solution of the nonlinear Schrödinger equation. © *1995 American Institute of Physics.*

1. The system of equations of nonlinear geometric optics [$\rho\geqslant 0$, $\alpha(\rho)>0$, and $\alpha(\rho)=4+\alpha_1\rho+\alpha_2\rho^2+\dots$ as $\rho\to 0$]

$$\rho_T+(\rho v)_X=0, \quad v_T+vv_X-\alpha(\rho)\rho_X=0 \tag{1}$$

is widely used for studying wave processes in unstable media.[1-4] This system differs from the equations of ordinary hydrodynamics by the sign of the pressure — here the pressure compresses the flow. The typical solutions, therefore, are not traveling waves but rather fragmentations into self-contracting bunches separated by empty intervals ($\rho=0$).[1] Other singularities, first investigated in Ref. 4, are also peculiar to the solutions of Eqs. (1).

In the present paper is concerned mainly on the local analysis of the process of collapse self-peaking, which is characteristic of pulses of the "general position." In Sec. 2 it will be shown that in this process the evolution of pulses near points of self-peaking is determined by the solution of a cubic equation corresponding to a cusp singularity which is universal for smooth mappings.[5] According to Eqs. (6)–(11) presented below, the collapse arises initially in the form of a point and then starts to expand. The amplitude of the pulse has a fold singularity on the two boundaries of the collapse[5]. As time elapses, one can expect some new collapses to occur, which also begin to expand, in complete agreement with the general tendency for separate clusters to form from the solutions (1).

It is obvious that during a collapse the smoothness of the intensity is lost only in the approximation of nonlinear geometric optics. In real situations the pulses remain smooth, and near the singular points of the solutions (1) they must be investigated in greater detail.

In Secs. 3–6 we show that when weak dispersion is taken into account, for a wide range of phenomena, a pulse near points of self-peaking is determined by a special isomonodromic solution of the nonlinear Schrödinger equation.

2. A distinguishing feature of our approach is its *local nature*. In contrast with the preceding studies, we do not use at all the explicit, completely determined solutions of the system (1). The only important point for us is that the system of equations of nonlinear geometric optics can be reduced locally, just as in hydrodynamics, to a linear system, treating T and X as coordinates, and ρ and v as independent variables:

$$X_\rho=\alpha(\rho)T_v+vT_\rho, \quad X_v=vT_v-T_\rho. \tag{2}$$

It is useful for us to reduce Eqs. (2), following, for example, Ref. 2 (p. 18), by means of the relations

$$T=B_v, \quad X=-B-\rho B_\rho+vB_v \tag{3}$$

to a single linear equation for $B(v,\rho)$:

$$\rho B_{\rho\rho}+2B_\rho+\alpha(\rho)B_{vv}=0. \tag{4}$$

The value $\rho=0$ is special for Eq. (4). The loss of the smoothness of the solutions (4) at the points $(v,0)$ could be associated with the vanishing of the intensity. Apparently, however, these losses of smoothness are either unimportant [ρ^{-1} is an exact solution of Eq. (4)] or they occur for infinite values of T and X. For this reason, we shall consider the intensity collapse processes which are determined only by the functions $B(v,\rho)$ which remain smooth in the limit $\rho\to 0$ and which, taking into account Eq. (4), can be expanded around $(v_*,0)$ in the following Taylor series:

$$\begin{aligned}B=&(T_*v_*-X_*)+T_*(v-v_*)+b_{01}[\rho-(v-v_*)^2/4]\\&+b_{11}(v-v_*)[\rho-(v-v_*)^2/12]+b_{02}\rho^2-(12b_{02}-\alpha_1 b_{01})(v-v_*)^2\rho/16\\&+b_{12}(v-v_*)\rho^2+b_{03}\rho^3+\dots\end{aligned} \tag{5}$$

[the points (T_*,X_*) correspond to the points $(v_*,0)$ in the hodograph plane].

If the Jacobian $J(v,\rho)=T_vX_\rho-T_\rho X_v=\alpha(\rho)(T_v)^2+\rho(T_\rho)^2$ does not vanish at the point $(v_*,0)$, then the coefficient $b_{01}=-2T_v(v_*,0)$ in the series (5) is different from zero. After making the change of variables

$$(T-T_*)=\tau, \quad (X-X_*)-(T-T_*)v_*=\xi \tag{6}$$

and substituting expansion (5) into Eqs. (3), we obtain the relations

$$\xi=-2b_{01}\rho+\dots, \quad \tau=-b_{01}(v-v_*)/2+b_{11}\rho+\dots . \tag{7}$$

The first relation in Eqs. (7) shows that the pulse amplitude $\rho^{1/2}$ in this case has a square-root singularity at the point $\xi=0$. These points constitute the entire *nonsingular* part of the line of the intensity collapse $T=T(X)$:

$$T=B_v(v,0), \quad X=-B(v,0)+vB_v(v,0). \tag{8}$$

The values $\xi/b_{01}>0$ obviously correspond to regions of zero intensity.

It is obvious, however, that during the evolution of initially smooth pulses the intensity collapses at the moment they appear can occur only at *separate* points. According to the relations (7), however, a necessary condition for the collapse points to be isolated is $b_{01}=J=0$ [on the collapse curve (8) these points are peaking points, since at these points $T_v=X_v=0$]. Substituting expansion (5) into Eq. (3) and using this equality, we obtain the relations

$$\xi-\tau(v-v_*)+b_{11}(v-v_*)[2\rho-(v-v_*)^2/12]+\ldots=0,$$

$$\tau-b_{11}[\rho-(v-v_*)^2/4]+\ldots=0, \tag{9}$$

which determine the evolution of the pulse at the moment an intensity collapse appears. Considering the situation of the "general position," we can assume that the coefficient b_{11} is different from zero. Under this assumption we have

$$\rho=\tau/b_{11}+(v-v_*)^2/4+\ldots, \tag{10}$$

and hence (since the intensity is non-negative) $b_{11}<0$. It follows from Eqs. (9) and (10) that near the points where a collapse first appears $v(T,X)$ is given approximately by the solution of the equation

$$\xi+\tau(v-v_*)+5b_{11}(v-v_*)^3/12=0, \tag{11}$$

which determines a cusp singularity.[5]

For small $\tau>0$ the quantity ρ therefore vanishes on the two curves which lie *outside* the region where solution (11) is not single-valued and which emerge from the point where the collapse appears. It is easy to see that on these curves the amplitude has a square-root singularity.

Remark 1. An example of the solution (2), for which it was calculated that when collapse appears, the intensity decreases as $\rho\approx(X-X_*)^{2/3}$ is presented in Ref. 1, (p. 104). A special corollary of Eqs. (10) and (11) is the conclusion that in the situation of the "general position" *self-peaking of a pulse occurs* according to this law at the precise moment of collapse. Of course, relations (7)–(11) are valuable mainly because they describe the dynamics of the collapse as a function of T.

3. We shall now examine the question of the effect of dispersion on the behavior of a pulse in the process of "collapse" in the approximation of nonlinear geometric optics for the equation

$$i\epsilon Q_T+\epsilon^2 Q_{XX}+K(|Q|^2)Q=0 \tag{12}$$

[$\epsilon\ll 1$, $K(0)=0$, $K'(y)=\alpha(y)/2$], which is often used to describe different nonlinear phenomena in unstable media.

The approximation of nonlinear geometric optics for the solutions of Eq. (12) is obtained by substituting into this equation the expression

$$Q=\rho^{1/2}\exp\{i\varphi/\epsilon\} \tag{13}$$

and dropping the terms of order $O(\epsilon^2)$: The system

$$\rho_T+2(\rho\varphi_X)_X=0, \quad \varphi_T+(\varphi_X)^2-K(\rho)=0 \tag{14}$$

is reduced to Eq. (1) with $v=2\varphi_X$ by differentiating the second equation with respect to X.

According to the method of matching,[6] the correct description of the solutions (12) near the points (T_*, X_*) of self-peaking of the intensity, which occurs in the nonlinear geometric optics approximation, is obtained after the scale transformation

$$\xi = x\epsilon^{3/4}, \quad \tau = t\epsilon^{1/2}, \quad Q = q\epsilon^{1/4}$$

[its form is determined by the requirements that all terms written out in Eqs. (10) and (11) be taken into account and the dispersion be balanced with the nonlinearity in Eq. (12)]. Equatiion (12) then becomes

$$i(q_t - v_*\epsilon^{-1/4}q_x) + q_{xx} + q(2|q|^2 + \epsilon^{1/2}\alpha_1|q|^4 + \ldots) = 0,$$

which, in turn, by means of the substitution

$$q = \exp\{i(-\epsilon^{1/2}(v_*)^2 t/4 + \epsilon^{-1/4}v_* x/2)\}p$$

reduces (in leading order in ϵ) to the nonlinear Schrödinger equation

$$ip_t + p_{xx} + 2|p|^2 p = 0 \tag{15}$$

which can be integrated by the method of the inverse problem of scattering.[7] Equation (15) must be supplemented by the condition that $p = b(t,x)\exp\{iS(t,x)\}$ match the nonlinear geometric optics approximation (13), which has at the point (T_*, X_*) a singularity described by Eqs. (10) and (11). We express this condition in terms of the solution of the cubic equation

$$x + 2tu + 10b_{11}u^3/3 = 0, \tag{16}$$

rewritten in the new notation from Eq. (11): As $x^2 + t^2 \to \infty$, the asymptotic relations

$$b \approx (u^2 + t/b_{11})^{1/2}, \quad S \approx \varphi(T_*, X_*)/\epsilon + [t^2/b_{11} + 5b_{11}u^2/6 + tu^2 + xu] + \ldots \tag{17}$$

should hold outside the region bounded by the two branches of the semicubic parabola

$$\tau^3 = (-9b_{11}x^2/16). \tag{18}$$

[In deriving the conditions (17), in addition to the completely obvious relation $S_x \approx u$, the second equation in (14) must be used.] These branches are the traces of the line (8), which separates the "bad" (Ref. 5, p. 32), "unstable" region of validity of the nonlinear geometric optics approximation from the "good," "stable" region of the collapse.

4. It turns out that the special solution of the nonlinear Schrödinger equation, which we are considering, has the following nonobvious property: Together with (15), it is a solution of the system of ordinary differential equations

$$p_{xxx} + 6|p|^2 p_x - (4tp_x - 2ixp)/\beta = 0 \quad (\beta = 4b_{11}/3), \tag{19}$$

$$p_{tt} = (4it/\beta + 2i|p|^2)p_t + (2i + 8t|p|^2)p/\beta + p_x(2ix/\beta + 2(p_x p^* - p_x^* p)),$$

$$(p_x)_t = 2xp/\beta + 4itp_x/\beta - 2i(p_x p^* - p_x^* p)p. \tag{20}$$

[The compatibility of Eqs. (19) and (20) with Eq. (15) is guaranteed by the fact that Eq. (19) is the sum of the stationary parts of the commutative and classical (Galilean-type) symmetries of the nonlinear Schrödinger equation.[8]]

It is remarkable that the same ordinary equations are also satisfied[9] by the special solution of the nonlinear Schrödinger equation from Ref. 10, which describes the behavior of the rapidly oscillating solutions, near the point of a caustic, of a series of dispersion equations with a small nonlinearity. However, these are nonetheless *different* solutions of Eqs. (15), (19), and (20) which correspond, on the basis of their origin, to the directly opposite physical situations.

For the decreasing solution from Ref. 10, as $x^2+t^2\to\infty$, dispersion predominates over nonlinearity. Conversely, for the increasing solution which we are studying, to leading order the dispersion can be neglected at infinity [outside the interior of the parabola (18)].

5. Substituting $p=b\exp\{iS\}$ into Eq. (15) and separating the real and imaginary parts of the result, we obtain after integrating the real part once the following system of fourth-order ordinary differential equations for the amplitude b and the derivative of the phase $f=S_x$:

$$b_{xx}b-(b_x)^2/2=3b^2[f^2-b^2]/2+2tb^2/\beta+\text{const},$$

$$f_{xx}b+3f_xb_x+3fb_{xx}=b[f^3+4tf/\beta-6fb^2-2x/\beta]. \tag{21}$$

Dropping the left-hand side of Eq. (21), which corresponds to the dispersion, we obtain the approximation $b^2\approx f^2+t/b_{11}$, $f\approx u$, which satisfies the condition (17). [The dispersion-free limit of the nonlinear Schrödinger equation (15) still must be taken into account.]

Remark 2. For $\alpha\equiv 4$ this approximation is the *exact* solution $\rho=\epsilon^{1/2}(u^2+t/b_{11})$, $v=v_*+\epsilon^{1/2}2u$ presented in Ref. 4 for the nonlinear geometric optics system (1). This fact, found by means of the "dispersionless" variant[11] of the symmetry approach to studying the nonlinear special functions of wave catastrophes,[12] served as a starting point for the entire investigation presented here.

6. The solution of the nonlinear Schrödinger equation and the solution from Ref. 10 belong to the class of isomonodromic solutions,[13] since together with the general equations of the method of the inverse problem of scattering[7] which are common to all solutions of Eq. (15)

$$\Psi_x\Psi^{-1}=U=-i\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+i\begin{pmatrix}0 & p\\ p^* & 0\end{pmatrix},$$

$$\Psi_t\Psi^{-1}=2\lambda U+i|p|^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-\begin{pmatrix}0 & p_x\\ -p_x^* & 0\end{pmatrix}, \tag{22}$$

the Ψ-function corresponding to them is the solution of an ordinary differential equation in the parameter λ:

$$\Psi_\lambda\Psi^{-1}=[-i(4\beta\lambda^3+4\lambda t-2\beta|p|^2\lambda+x)-\beta(p_xp^*-p_x^*p)]\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
$$+4i\beta\lambda^2\begin{pmatrix}0 & p\\ p^* & 0\end{pmatrix}-2\beta\lambda\begin{pmatrix}0 & p_x\\ -p_x^* & 0\end{pmatrix}+\begin{pmatrix}0 & 4itp-\beta p_t\\ 4itp^*+\beta p_t^* & 0\end{pmatrix}. \tag{23}$$

Since for each sector $S_j=\{\lambda:\pi(j-1)/4<\arg\lambda<\pi(j+1)/4\} j=1,\dots,8$ there exists (Ref. 14, p. 66) a solution Ψ_j of the system (23), which in the limit $\lambda\to\infty$ has in this sector the asymptotic form

$$\Psi_j\approx\exp\left\{[-i(\lambda x+2t\lambda^2+\beta\lambda^4)+\gamma\ln\lambda]\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\},$$

our solution of the nonlinear Schrödinger equation, just as the solution from Ref. 10, can be treated as an analog of the Pearcey integral $A=\int_R\exp\{-2i(x\lambda+2t\lambda^2+\beta\lambda^4)\}d\lambda$. [This interpretation is also natural because $A(t,x)$ is the exact solution of the linear part of Eqs. (15), (19), and (20)].

Since Eq. (23) is consistent with Eq. (22), the Stokes matrices $M_k=(\Psi_k)^{-1}\Psi_{k+1}$ do not depend on t and x. This invariance of M_k can be used to clarify the behavior of p in the limit $x^2+t^2\to\infty$ and in the "good" region, lying inside the curve (18), as well as in a neighborhood of the curve. As far as the last problem is concerned, it can be expected, on the basis of one of the analogies with Ref. 9, that here p should be given with the aid of a solution of an ordinary differential equation which has the form of the second Painlevé equation $g_{zz}=zg-2|g|^2g$, which is a nonlinear generalization of the classical Airy function. In contrast to the solution studied in Ref. 9, however, in the limit $z\to\infty$ its amplitude should increase as $|g|\approx z^{1/2}$. (This solution corresponds to Ref. 15 .) In general, it should play a key role in the description of the fast transition from the quasiclassical regime to the region of "zero" intensity, which occurs in neighborhoods [of width of the order of $O(\epsilon^{2/3})$] of the branches of the curve (8). In the region of "zero" intensity it is most natural to expect high-frequency oscillations of Q with the amplitude $O(\epsilon^{1/2})$. [However, exact results concerning the question of taking into account the dispersion outside small neighborhoods of the points (X_*,T_*) have not yet been obtained. This question requires further study.]

In conclusion, we give one more property of p. Since p is obviously an even function of x, on the straight line $x=0$ (*Maxwell's stratum*[16] of the cusp catastrophe) Eq. (20) reduces to the fourth Painlevé equation.[17] This fact confirms the special role, pointed out in Ref. 18, which Maxwell's strata play in the special functions of wave catastrophes.

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[1] S. K. Zhdanov and B. A. Trubnikov, *Quasigaseous Unstable Media* [in Russian], Nauka, Moscow, 1991.
[2] A. B. Shvartsburg, *Nonlinear Geometric Optics in the Nonlinear Theory of Waves* [in Russian], Nauka, Moscow, 1976.
[3] K. A. Naugol'nykh and L. A. Ostrovskiĭ, *Nonlinear Wave Processes in Acoustics* [in Russian], Nauka, Moscow, 1990.
[4] A. V. Gurevich and A. B. Shvartsburg, Zh. Éksp. Teor. Fiz. **58**, 2012 (1970) [Sov. Phys. JETP **31**, 1084 (1970)].
[5] V. I. Arnol'd, *Catastrophe Theory* [in Russian], Nauka, Moscow, 1990.
[6] A. M. Il'in, *Matching of Asymptotic Expansions of the Solutions of Boundary-Value Problems* [in Russian], Nauka, Moscow, 1989.

[7] V. E. Zakharov and A. B. Shabat, Zh. Éksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[8] A. Yu. Orlov and E. I. Shul'man, Teor. Matem. Fiz. **64**, 323 (1985).
[9] B. I. Suleĭmanov, Zapiski LOMI **187**, 110 (1991).
[10] R. Haberman and R. Sun, Studies in Appl. Math. **72**, 39 (1985).
[11] V. R. Kudashev, "KdV shock-like waves as invariant solutions of KdV equation symmetries," Preprint, Los Alamos Nonlinear Science PBB patt-soll/9404002, 1994; Integriruemost' v dinamicheskikh sistemakh [Integrability in Dynamical Systems], Ufa, **70**, 1994.
[12] B. I. Suleĭmanov and I. T. Khabibullin, Teor. Matem. Fiz. **97**, 213 (1993).
[13] A. R. Its, Izv. Akad. Nauk SSSR, Ser. Matem. **49**, 530 (1985).
[14] W. Wazow, *Asymptotic Expansions for Ordinary Differential Equations*, Interscience Publishers, N. Y., 1965..
[15] A. A. Kapaev, Teor. Matem. Fiz. **77**, 323 (1988).
[16] V. I. Arnol'd, Usp. Matem. Nauk **47**, 3 (1993).
[17] A. V. Kitaev, J. Math. Phys. **35**, 2934 (1994).
[18] B. I. Suleĭmanov, Zh. Éksp. Teor. Fiz. **104**, 1089 (1994) [JETP **78**, 583 (1994)].

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