Unification of all string models with $c<1$ in the framework of generalized Kontsevich model

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A 1-matrix model which nicely interpolates between double-scaling continuum limits of all multimatrix models, is proposed. The interpolating partition function is always a KP τ-function, and always obeys $\mathcal{L}_{-1}$ constraint and string equation. This model can therefore be considered as a natural unification of all models of 2D gravity (string models) with $c\leqslant 1$.

The model. The purpose of this letter is to introduce a new theory, which we call *Generalized Kontsevich's Model* (GKM) and to describe its structure and appealing properties. The partition function of the GKM is defined by the following integral over $N\times N$ Hermitian matrix:

$$Z_N^{\{\mathcal{V}\}}[M] \equiv \frac{\int e^{U(M,Y)}dY}{\int e^{-U_2(M,Y)}dY}, \tag{1}$$

where

$$U(M,Y) = Tr[\mathcal{V}(M+Y) - \mathcal{V}(M) - \mathcal{V}'(M)Y] \tag{2}$$

and

$$U_2(M,Y) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}U(M,\epsilon Y) \tag{3}$$

is a Y^2 term in U. M is also a Hermitian $N\times N$ matrix with eigenvalues $\{\mu_i\}$, and $\nu(\mu)$ is arbitrary analytic function.

Integrable structure. After the shift of variables $X=Y+M$ and integration over angular components of X, $Z_N^{\{\nu\}}[M]$ acquires the form of

$$Z_N^{\{\mathcal{V}\}}[M] = \frac{[\det\tilde{\Phi}_i(\mu_j)]}{\Delta(M)}, \tag{4}$$

where $\Delta(M) = \prod_{i<j}(\mu_i-\mu_j)$ is the Van der Monde determinant, and the functions

$$\tilde{\Phi}_i(\mu) = [\mathcal{V}''(\mu)]^{1/2} e^{\mathcal{V}(\mu)-\mu\mathcal{V}'(\mu)} \int e^{-\mathcal{V}(x)+x\mathcal{V}'(\mu)} x^i dx \ . \tag{5}$$

The only assumption necessary for the derivation of (4) from (1) is that the potential $\nu(\mu)$ can be represented as a formal series in positive *integer* powers of μ.

Equation (4) with *arbitrary* entries $\phi_i(\mu)$ is characteristic for generic KP τ-function $\tau^G(T_n)$ in Miwa's coordinates

$$T_n = \frac{1}{n}\mathrm{Tr} M^{-n}, \quad n \geq 1 \tag{6}$$

and the point G of Grassmannian is defined by potential ν through the set of basis vectors $\{\phi_i(\mu)\}$. (We recall that *a priori* definition is $\tau^G(T_n) = \langle 0|e\Sigma^{T_n J_n} G|0\rangle$, where J stands for the free-fermion $U(1)$ current and G is an exponent of quadratic combination of free fermion operators.) Therefore,

$$Z^{\{\mathcal{V}\}}[M] = \tau^{\{\mathcal{V}\}}(T_n). \tag{7}$$

The case of finite N in this formalism is distinguished by the condition that only N of the parameters $\{\mu_i\}$ are finite. In order to take the limit $N \to \infty$ in the GKM(1), it is enough to bring all the $\mu_i' \cdot s$ from infinity. In this sense this is a smooth limit in contrast with the singular conventional double-scaling limit, which must be taken into account in ordinary (multi) Matrix models.

$\mathscr{L}_{-1}$ **constraint.** The set of functions $\{\tilde{\Phi}_i(\mu)\}$ in (4) is, however, not arbitrary. They are all expressed in terms of a single function—the potential $\nu(\mu)$—and are in fact recurrently related: if we denote the integral in (5) through $F_i(\nu'(\mu))$, then

$$F_i(\lambda) = (\partial/\partial\lambda)^{i-1} F_1(\lambda). \tag{8}$$

This relation is enough to prove that

$$\frac{\partial}{\partial T_1} \log\ Z_N^{\{\mathcal{V}\}} = \mathrm{Tr} M - \mathrm{Tr}\frac{\partial}{\partial \Lambda_{tr}} \log\det F_i(\lambda_j) \tag{9}$$

when the potential $\nu(\mu)$ increases faster than μ as $\mu \to \infty$.

Thus, $Z^{\{\nu\}}$ satisfies a simple identity:

$$\frac{1}{Z^{\{\mathcal{V}\}}} \mathcal{L}_{-1}^{\{\mathcal{V}\}} Z_N^{\{\mathcal{V}\}} = \frac{\partial}{\partial T_1} \log Z_N^{\{\mathcal{V}\}} - \mathrm{Tr} M + \mathrm{Tr}\frac{\partial}{\partial \Lambda_{tr}} \log\det F_i(\lambda_j) = 0\ , \tag{10}$$

where the operator $\mathscr{L}_{-1}^{\{\nu\}}$ is defined as follows:

$$\mathcal{L}_{-1}^{\{\mathcal{V}\}} = \sum_{n\geq 1} \mathrm{Tr}\left[\frac{1}{\mathcal{V}''(M) M^{n+1}}\right] \frac{\partial}{\partial T_n}$$

$$+\frac{1}{2}\sum_{i,j} \frac{1}{\mathcal{V}''(\mu_i)\mathcal{V}''(\mu_j)} \frac{\mathcal{V}''(\mu_i) - \mathcal{V}''(\mu_j)}{\mu_i - \mu_j} - \frac{\partial}{\partial T_1} \tag{11}$$

(the items with $i=j$ are included into the sum). The reason why this operator is denoted by $\mathscr{L}_{-1}$ will be clear after the reductions of GKM will be discussed. From Eqs. (9) and (10) it follows that the partition function of GKM usually satisfies the constraint

$$\mathcal{L}_{-1}^{\{\mathcal{V}\}}\tau^{\{\mathcal{V}\}}=0. \tag{12}$$

Reductions. The integral $\mathscr{F}^{\{v\}}[\Lambda]$, $\Lambda\equiv v'(M)$, in the numerator of (1) satisfies the Ward identity

$$\mathrm{Tr}\left\{\epsilon(\Lambda)\left[\mathcal{V}'\left(\frac{\partial}{\partial\Lambda_{\mathrm{tr}}}\right)-\Lambda\right]\right\}\mathcal{F}_N^{\{\mathcal{V}\}}=0 \tag{13}$$

[as result of invariance under any shift of integration variable $X\to X+\epsilon(M)$]. If the potential $v(\mu)$ is restricted to be a polynomial of degree $K+1$, this identity will imply that the functions (8) obey additional relations:

$$F_{m+Kn}(\lambda)=\lambda^n\cdot F_m(\lambda)+\sum_{i=1}^{m+Kn-1}s_iF_i(\lambda). \tag{14}$$

Since the sum at the *r.h.s.* does not contribute to determinant (5), we can say that all the functions F_n are expressed in terms of the first K functions $F_1...F_K$ by multiplication by powers of $\lambda=v'(\mu)$. Such a situation (when the basis vectors ϕ_i, which define the point of Grassmannian, are proportional to the first K ones) corresponds to a reduction of the KP hierarchy. This reduction depends on the form of $v'(\mu)$ and in the case $v(\mu)=v_K(\mu)=$ const $\cdot\mu^{K+1}$ coincides with the well-known K reduction of the KP hierarchy (KdV as $K=2$, Boussinesq as $K=3$, etc.). Thus in such cases the partition function of GKM becomes the $\tau^{\{K\}}$ function of the corresponding hierarchy. Moreover, in this case one can represent each function $\widetilde{\Phi}_i(\mu)$ in (5) in the form of $A^{i-1}\widetilde{\Phi}_1(\mu)$, where A is a proper differential operator which can be read out from Eqs. (5) and (8). Surprisingly, this operator coincides with that of Ref. 1, which implements a different approach based on transmuting the Virasoro ($\mathscr{W}$-) constraints into the conditions on the point of Grassmannian.

Generic $\tau^{\{K\}}$ possesses an important property; it is almost independent of all the time variables T_{nK}. To be exact,

$$\partial\log\tau^{\{K\}}/\partial T_{nK}=a_n=\mathrm{const}\,. \tag{15}$$

If $v=v_K$, the generic expression (12) for the $\mathscr{L}_{-1}$ operator becomes

$$\mathcal{L}_{-1}^{\{K\}}=\frac{1}{K}\sum_{n>K}nT_n\partial/\partial T_{n-K}+\frac{1}{2K}\sum_{\substack{a+b=K\\a,b>0}}aT_abT_b-\partial/\partial T_1\,. \tag{16}$$

The last item at the *r.h.s.* may be eliminated by the shift of the time variables:

$$T_n\to\hat{T}_n^{\{K\}}=T_n-\frac{K}{n}\delta_{n,K+1}. \tag{17}$$

This shift is, however, K-dependent and does not seem to have too much sense. However, if it is expressed only in terms of these $\widehat{T}$'s, the constraint (12) acquires the form

$$\mathcal{L}_{-1}^{\{K\}}\tau^{\{K\}} = \left\{ \frac{1}{K} \sum_{\substack{n>K \\ n\neq 0 \bmod K}} n\hat{T}_n \partial/\partial\hat{T}_{n-K} + \frac{1}{2K} \sum_{\substack{a+b=K \\ a,b>0}} a\hat{T}_a b\hat{T}_b \right\} \tau^{\{K\}}$$

$$= \sum_n a_n (n+1)\hat{T}_{(n+1)K}\tau^{\{K\}}, \tag{18}$$

where the *l.h.s.* was taken from Ref. 2. The sum at the *r.h.s.* in (18) does not contribute to the "string equation"

$$\frac{\partial}{\partial T_1} \frac{\mathcal{L}_{-1}^{\{K\}}\tau^{\{K\}}}{\tau^{\{K\}}} = 0. \tag{19}$$

Moreover, in the variance with generic $\tau^{\{K\}}$ the partition function $Z^{\{K\}}$ of GKM is expected to obey (15) and (18) with all $a_n = 0$.

Universal string equation. Generalization of (19) to the case of arbitrary potential

$$\frac{\partial}{\partial T_1} \frac{\mathcal{L}_{-1}^{\{\mathcal{V}\}}\tau^{\{\mathcal{V}\}}}{\tau^{\{\mathcal{V}\}}} = 0 \tag{20}$$

can be transformed to the form

$$\sum_{n\geq -1} \mathcal{T}_n \frac{\partial^2 \log \tau}{\partial T_1 \partial T_n} = u, \tag{21}$$

where

$$\mathcal{T}_n \equiv Tr \frac{1}{\mathcal{V}''(\mathcal{M})} \frac{1}{\mathcal{M}^{n+1}}, \tag{22}$$

$$u \equiv \frac{\partial^2 \log \tau}{(\partial T_1)^2}, \quad \frac{\partial \log \tau}{\partial T_0} \equiv 0, \quad \frac{\partial \log \tau}{\partial T_{-1}} \equiv T_1.$$

If Baker–Akhiezer are introduced

$$\Psi_{\pm}(z|T_k) = e^{\sum T_k z^k} \frac{\tau(T_n \pm \frac{z^{-n}}{n})}{\tau(T_n)}, \tag{23}$$

the string equation (22) can be rewritten in the form of bilinear relation

$$\sum_i \frac{\Psi_+(\mu_i)\Psi_-(\mu_i)}{\mathcal{V}''(\mu_i)} = u. \tag{24}$$

$\mathscr{W}$ **constraints.** According to the arguments of Ref. 2, the constraint

$$\mathcal{L}_{-1}^{\{K\}}\tau^{\{K\}}=0 \tag{25}$$

[i.e., (18) with the vanishing *r.h.s.*, as it is in fact the case if we deal with model (1)] implies the entire tower of $\mathscr{W}$ constraints

$$\mathcal{W}_{Kn}^{(k)}Z^{\{K\}}=0, \quad k=2,3,...,K; \quad n\geq 1-k \tag{26}$$

imposed on $\tau^{\{K\}}$. Here $\mathscr{W}_{Kn}^{(p)}$ is the nth harmonic of the pth generator of Zamolodchikov's W_K algebra (the proper notation would be $\mathscr{W}_n^{(p)\{K\}}$, but it is rather complicated). There is a Virasoro–Lie subalgebra, which is generated by $\mathscr{W}_{Kn}^{(2)}=L_n^{\{K\}}$, and the particular $\mathscr{L}_{-1}^{\{K\}}$ is the operator (16). This is the origin of our notation $\mathscr{L}_{-1}^{\{v\}}$ in the generic situation (where the entire Virasoro subalgebra of W_∞ was not explicitly specified).

In addition to being a corollary of (24), the constraints (25) can be directly deduced from the Ward identity (13). For the case $K=2$ (which is original Kontsevich's model[3]) this derivation was given in Ref. 4 (see also Refs. 5 and 6 for alternative proofs). Unfortunately, for $K\geqslant 3$ the direct corollary of (13) is not just (25), but peculiar linear combinations of these constraints, e.g., for $K=3$ they are

$$\mathcal{W}_{3n}^{(3)}Z_\infty^{\{3\}}=0, \; n\geq -2;$$

$$\left\{\sum_{k\geq 1}(3k-1)\hat{T}_{3k-1}\mathcal{W}_{3k+3n}^{(2)}+\sum_{a+b=3n}\frac{\partial}{\partial T_{3a+2}}\mathcal{W}_{3b-3}^{(2)}\right\}Z_\infty^{\{3\}}=0,$$

$$a,b\geq 0, \; n\geq -2;$$

$$\left\{\sum_{k\geq 1+\delta_{n+3,0}}(3k-2)\hat{T}_{3k-2}\mathcal{W}_{3k+3n}^{(2)}+\sum_{a+b=3n}\frac{\partial}{\partial T_{3a+1}}\mathcal{W}_{3b-3}^{(2)}\right\}Z_\infty^{\{3\}}=0, \; a,b\geq 0, \; n\geq -3. \tag{27}$$

For identification of (26) with (25) one can argue that both sets of constraints possess unique, and thus coinciding, solutions.

Multimatrix models. While detailed investigation of the properties of multimatrix models in the double-scaling limit (the analog of Ref. 7 in the case of conventional Hermitian 1-matrix model) is still lacking, it was suggested in Ref. 2 that the square roots of their partition functions $\sqrt{\Gamma_{ds}^{\{K-1\}}}$ ($K-1$ is the number of matrices, index ds means that the partition function is considered in the double scaling limit), have the following properties:

$$\mathcal{W}_{Kn}^{(k)}\sqrt{\Gamma_{ds}^{\{K-1\}}}=0 \quad , \quad k=2,3,...,K; \quad n\geq 1-k. \tag{28}$$

Comparing these properties to the above information about GKM, we obtain

$$Z_{\infty}^{\{K\}} = \sqrt{\Gamma_{ds}^{\{K-1\}}}\ . \tag{29}$$

Conclusion. We presented a brief description of the properties of the GKM, which is defined by Eq. (1). Its partition function may be considered as a functional of two different variables: potential $\nu(\mu)$ and the infinite-dimensional Hermitian matrix M with the eigenvalues $\{\mu_i\}$. The partition function $Z_N^{\{\nu\}}$ is an N-independent KP τ-function, which is considered as a function of time-variables $T_n = \frac{1}{n} TrM^{-n}$, and the point of Grassmannian is specified by the choice of the potential. The N-dependence enters only through the argument M: we return to the finite-dimensional matrices if only N eigenvalues of M are finite. In this sense, the "continuum" limit of $N \to \infty$ is smooth.

The GKM is associated with a subset of Grassmannian, specified by additional $\mathscr{L}_{-1}$ constraint (12). For well-adjusted potentials $\nu(\mu) =$ const $\cdot \mu^{K+1}$, the corresponding points in Grassmannian lies in the subvarieties, associated with K-reductions of KP hierarchy, $Z^{\{\nu\}}$ becomes independent of all the time variables T_{Kn}, and the $\mathscr{L}_{-1}$ constraint implies the whole tower of W_K-algebra constraints on the reduced τ-function. These properties are exactly the same as suggested for double scaling limit of the $K-1$-matrix model, and in fact there is an identification (29).

All this means that GKM provides an interpolation between the double-scaling continuum limits of all multimatrix models and thus between all string models with $c \leqslant 1$. Moreover, this is a reasonable interpolation, because both integrable and "string-equation" structures are preserved. This is why we advance GKM as a plausible (on-shell) prototype of a unified theory of 2D gravity. All the proofs will be presented in Ref. 8.

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