Contour dynamics of the Hasegawa–Mima equation

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The contour dynamics of the Hasegawa–Mima equation is completely integrable if the dimensions of the contour are large in comparison with the Rossby radius.

1. Contour dynamics is widely used in 2D hydrodynamics.[1–3] This method, which was introduced in Ref. 1, can generally be formulated as follows: Assume that a 2D or quasi-2D motion of a liquid is controlled by the equation

$$(\partial_t + [\nabla\psi, \nabla])\Gamma = 0, \tag{1}$$

where ψ is the stream function, Γ is a generalized vorticity, and [,] means the z component of the vector product. The vorticity Γ is expressed in terms of ψ in some manner. For the 2D Euler equation, for example, this expression is $\Gamma = \Delta\psi$, while for the Hasegawa–Mima equation[14] it is

$$\Gamma = \Delta\psi - \psi/a^2, \tag{2}$$

where a is the Rossby radius. More complex relationships are also possible [in Ref. 15, the relationship $\Gamma = -\psi/a^2 - (1 - \exp(\rho^2\Delta))I_0(-\rho^2\Delta)\psi/\rho^2$, where I_0 is the modified Bessel function, was studied].

Let us assume that the vorticity Γ initially has a value Γ_0 in some region Ω and is zero outside Ω. As time elapses, the region Ω will then be distorted in some manner, but the vorticity will remain concentrated in Ω and equal to Γ_0. Consequently, the $(1+2)$-dimensional equation in (1) becomes, under these initial conditions, a $(1+1)$-dimensional equation describing the motion of the boundary $\gamma = \partial\Omega$. The

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normal component of the velocity (we need concern ourselves with only this component) is

$$v_n = \psi_s, \tag{3}$$

where s is the distance along γ. Consequently, contour dynamics is defined if we have expressed ψ on γ as some functional of γ (which depends on Γ_0 and on the position on γ). Contour dynamics is usually nonlocal.

We see that for the Hasegawa–Mima equation in the case of large contours (with a characteristic length much greater than a) contour dynamics becomes local [as follows in a completely natural way from expression (2)] and is completely integrable (a pleasant surprise).

2. For the Hasegawa–Mima equation, ψ is expressed in terms of Γ by means of the modified Bessel function K_0;

$$\psi(R) = -(\Gamma_0/2\pi)\int_\Omega d^2 r K_0(|\vec{r}-\vec{R}|/a). \tag{4}$$

Making the substitution $K_0 = a^2\Delta K_0$ (for $\vec{R}$ outside Ω), and going through the obvious manipulations, we find

$$\psi(s) = (\Gamma_0 a/2\pi)\int ds' K_1(|\vec{r}(s')-\vec{r}(s)|/a)|\vec{r}(s')-\vec{r}(s)|^{-1} \times\{(x(s')-x(s))y_s(s')-(y(s')-y(s))x_s(s')\}, \tag{5}$$

where $\vec{r}(s) = \{x(s),y(s)\}$ is a vector which runs along γ. For large contours ($a \ll r_{\text{char}}$), expression (5) reduces to the local expression

$$\psi = (\Gamma_0 a^3/4)K, \tag{6}$$

where $K = x_s y_{ss} - x_{ss} y_s$ is the curvature. In dimensionless form, the contour dynamics of interest here is determined by the velocity

$$v_n = K_s. \tag{7}$$

Relation (7) conserves the length of the contour (area is conserved in any contour dynamics).

3. The evolution equation for γ which is determined by (7) is

$$x_t = -K_s y_s + E x_s,$$

$$y_t = K_s x_s + E y_s, \tag{8}$$

where E is determined from the requirement $x_s^2 + y_s^2 = 1$. Simple manipulations lead to $E = K^2/2$. After some further transformations, we find an evolution equation for the curvature:

$$K_t = K_{sss} + (3/2)K^2 K_s. \tag{9}$$

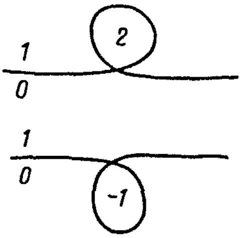

FIG. 1. The solitons in (10) with different signs of κ.

Equation (9) is the standard, completely integrable, modified Korteweg-de Vries equation.[16,17] Its soliton is

$$K = 2\cosh^{-1}(\kappa(s - \kappa^2 t)). \tag{10}$$

The angle at which the tangent vector $\vec{r}_s$ is oriented is found by integrating K. For $t = 0$ we find

$$\alpha = 2\arctan(\sinh(\kappa s)) + \pi \text{sign}(\kappa). \tag{11}$$

Solitons with opposite signs are shown in Fig. 1.

4. We conclude with a list of questions and suggestions that require study.

A. The soliton in Fig. 1 is a loop and thus requires three different values of the vorticity, rather than the two assumed. In addition, the process represented in Fig. 2 is permissible for (9) but obviously impossible for (1). The reason is that (6) is equivalent to (5) only as long as γ does not touch itself. If we wish to incorporate such cases in our scheme, we need to go through a reconnection procedure of the contour surgery type.[10]

B. If a self-intersecting contour γ is a steady-state contour by virtue of (9), one might suggest that the Hasegawa–Mima equation would have similar steady states, i.e., V states, at sufficiently large values of a (Refs. 2, 4, and 6). Here the steady-state modified Korteweg–de Vries equation in (9), with periodic boundary conditions, can be used to search for V states. Clearly, the soliton of Fig. 1 will survive in one form or another. Some extremely exotic vortex formations of the tripole type were observed in the experiments of Ref. 18 (admittedly, the physical conditions in Ref. 18 were not the same).

C. The contour dynamics in (7) may also prove useful in the theory of completely

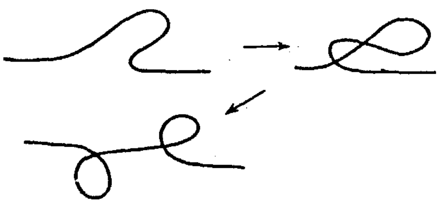

FIG. 2. Dispersal of solitons of different signs.

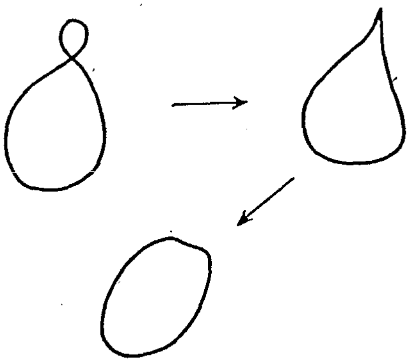

FIG. 3. A transformation of a loop with zero circuits (a figure-eight) into a loop with one circuit is not possible unless there is a singularity in one stage.[19]

integrable systems. Let us consider some examples. 1. The first invariant has the same value for all the solitons [this value is $\pm 2\pi$ in normalization (9)]. These solitons are therefore simply loops. 2. The first invariant in (9)—an integral of the curvature—is the only invariant of smooth transformations of closed curves in a plane.[19] It can be shown that the process illustrated in Fig. 3 is not possible in contour dynamics (7). The reason is that (7) conserves the smoothness of contours. 3. Figure 2 may be regarded as an illustration of the separation of solitons of different signs.

D. If γ is given in the form $\gamma = f(x)$, the evolution equation for f corresponding to (7) can be written

$$u_t = (1 - u^2)^{3/2} u_{xxx}, \tag{12}$$

where $u = f_z (1 + f_x^2)^{-1/2}$. Equation (12) does not have any solitons. If, on the other hand, we allow multivalued functions f, and if we make the nontrivial change of variables $(x,u) \to (s,K)$, where

$$s = \int dx((1 + f_x^2)^{1/2} - 1), \quad K = u_x,$$

then Eq. (12) reduces to the modified Korteweg–de Vries equation.

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After this paper had been submitted for publication, L. J. Pratt of Woods Hole informed me that he, too, had derived Eq. (7), in Ref. 20. A further analysis of this equation, including some of the results which we found, is to be published in Ref. 21.

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