Charge-transport statistics in quantum conductors

L. S. Levitov[1),2)] and G. B. Lesovik[4),3)]

[1)] *L. D. Landau Institute of Theoretical Physics, 117334, Moscow;* [2)] *Massachusetts Institute of Technology;* [3)] *Institute of Solid State Physics, 142432, Chernogolovka, Moscow Oblast;* [4)] *Universität zu Köln, Institut für Theoretische Physik*

(Submitted 9 April 1992)

Pis'ma Zh. Eksp. Teor. Fiz. **55**, No. 9, 534–537 (10 May 1992)

The noise in a quantum resistor at $T=0$ is described statistically over a long time interval $t \gg \tau_V = \hbar/eV$. The correlation functions of all orders are found. The distribution function of the transmitted charge is also found. The noise is determined by negative current pulses with a length shorter than τ_V. They carry a charge which is a multiple of $e^* = 2e\sqrt{D}$, where D is the transparency. The statistics of the pulses is binomial. This result corresponds to the probability distribution which arises as the result of a steady-state Bernoulli random process.

The noise observed in microscopic resistors at low temperatures is in excess of the equilibrium Nyquist noise.[1] This noise is made up of several components: the $1/f$ noise, the random telegraph signal from the switching of defects,[2] and the current fluctuations which stem from the discrete nature of the charge.[3,4] The latter component, also known as "quantum shot noise," is our topic in the present letter. If only one conduction channel is effective, the spectral density of the latter noise component is[3]

$$S_V(\omega=0) = \frac{e^2}{\pi^2\hbar} D(1-D)eV, \tag{1}$$

where D is the transparency (the square of the transmission amplitude), and V is the voltage. Expression (1) can also be generalized to the case in which more than one channel is operating.[5] It is worthwhile to analyze the statistical properties of this noise in more detail, in order to (for example) make a comparison with the statistics of classical shot noise. Our purpose in the present letter is to derive expressions like (1) for the high-order correlation functions of the current. In the one-channel case we find the correlation functions of *all* orders exactly. Using them, we find a complete description of the statistics of the charge fluctuations. The primary result of this letter is that the statistics is binomial, with probabilities $q, p = \frac{1}{2}(1 \pm \sqrt{D})$ and with a *noninteger* quantum of charge, namely, $2e\sqrt{D}$.

The problem is formulated as follows.[3] A one-channel Landauer resistor[6] can be thought of as a one-dimensional potential barrier on which electrons are incident from both sides, coming from reservoirs on the left and right. The energy distributions in these reservoirs are equilibrium Fermi distributions: $n_L(E) = \theta(eV/2 - E)$, $n_R(E) = \theta(-eV/2 - E)$ (we are assuming $T=0$). The potential difference gives rise to a current $I = 2(e^2/h)DV$ in the reservoirs. (For simplicity we ignore Coulomb screening,[6] and we furthermore assume that the transparency is independent of E. This assumption is valid if $eV \ll \Delta E$, where the right side is a characteristic energy scale of the changes in D.) The reasons for the charge fluctuations are the Fermi statistics of the electrons and the operator nature of the current in the quantum-mechanical problem.

We are interested in the charge $\hat{Q}_t = \int_0^t \hat{I}(x,t')dt'$ which has passed over a time $t \gg \tau_V = \hbar/eV$. We write the quantities $\hat{Q}_t$ and $\hat{I}(x,t)$ in terms of the operators a_p, a_p^+, b_p, b_p^+, which annihilate and create left-hand and right-hand states.[3] We calculate expectation values of the type $\langle \hat{I}(x,t_1)...\hat{I}(x,t_k) \rangle$ by the standard procedure for statistical averaging:[7] (a) We form links of operator pairs, a_p, a_p^+ and b_p, b_p^+, with which we associate the expectation values $\langle a_p^+ a_{p'} \rangle = n_L(E_p)\delta(p-p')$, and $\langle a_p a_{p'}^+ \rangle = [1-n_L(E_p)]\delta(p-p')$, (and we do the corresponding thing for b_p, b_p^+). (b) We determine the sign by making use of the circumstance that the operators anticommute. (c) We sum over all ways to arrange the links. We find the expectation values $\langle \hat{Q}_t^k \rangle$ by evaluating the integral $\int_0^t ... \int_0^t \langle \hat{I}_1 ... \hat{I}_k \rangle dt_1 ... dt_k$. As usual, the result found in this averaging can be formulated most simply in terms of irreducible correlation functions $\langle\langle \hat{Q}_t^k \rangle\rangle$, which are given by graphs in which the links form a single closed loop. The integration over time gives rise to a factor t on each such loop. The momenta p on all links in a loop become identical, and we are left with a single integration over (dp) on a loop. In this case the dependence on the point x at which the current I is calculated drops out of the picture. Furthermore, it is not difficult to see that only states with $(-1/2)eV < E_p < (1/2)eV$ make a nonzero contribution to the expectation value. We thus find another factor of $2(eV/2\pi h)$ after the integration over p (the 2 arises from the summation over spins). We thus find $N = (2e/h)Vt$ on each loop.

The factor N represents everything which depends on the integration over dt and dp. Using this simple fact, we can formulate the averaging rules in a slightly different manner, avoiding from the outset the coordinate dependence and the time dependence of the expressions to be averaged. We consider the operator

$$\hat{J} = e\sum_{i=1}^{N} Da_i^+ a_i - Db_i^+ b_i + \Lambda a_i^+ b_i + \Lambda b_i^+ a_i \quad , \tag{2}$$

where a_i, b_i are fermion operators; $D = A^2$; $\Lambda = AB$; and A and B are the transmission and reflection amplitudes of the barrier $(A^2 + B^2 = 1)$. For simplicity we assume $A^* = A$, $B^* = B$; it is simple to verify that all the results remain the same if A and B are instead complex. The operator $\hat{J}$ is the current operator[3] $\hat{I}(x,t)$ after the coordinate dependence and the field dependence have been removed. In it, states from the band $|E_p| < \frac{1}{2}eV$, are singled out; only these states contribute to the expectation values of interest here. It follows that we have $\langle \hat{Q}_t^k \rangle = \langle \hat{J}^k \rangle$. The averaging of J^k should be carried out in accordance with the rules $\langle a_i^+ a_j \rangle = \langle b_i b_j^+ \rangle = \delta_{ij}$, $\langle a_i a_j^+ \rangle = \langle b_i^+ b_j \rangle = 0$. Strictly speaking, calculations with operator (2) are meaningful only at integer values of N, but at $N \gg 1$ the result can be continued analytically from integer to noninteger values of N.

With these rules formulated, finding the expectation value of $\hat{Q}_t^k$ of any order reduces to the mechanical collection of terms of the form $\pm N^L e^k D^m \Lambda^{k-m}$, where L is the number of closed loops. Here are the results for the first four irreducible correlation functions:

$$\langle\langle \hat{Q}_t \rangle\rangle = eDN, \quad \langle\langle \hat{Q}_t^2 \rangle\rangle = e^2 D(1-D)N, \quad \langle\langle \hat{Q}_t^3 \rangle\rangle = -2e^3 D^2(1-D)N,$$

$$\langle\langle \hat{Q}_t^4 \rangle\rangle = 2e^4 D^2(1-D)(3D-1)N, \quad N = \frac{2e}{h} Vt. \tag{3}$$

Here we have used the relation $D^2 + \Lambda^2 = D$. The first correlation function is simply the Landauer formula, and the second is a known result [see (1)]. The fact that the correlation functions of third and fourth orders are nonzero means that the noise is not Gaussian. We note that the correlation function of third order is negative, while that of fourth order changes sign at $D = 1/3$.

We can now calculate the characteristic function $\chi(\lambda) = \langle \exp(-i\lambda \hat{Q}_t) \rangle$. Once we have found $\chi(\lambda)$, we can find all the irreducible correlation functions by using the well-known expansion[8]

$$ln(\chi(\lambda)) = \sum_{k=1}^{\infty} \frac{(-i\lambda)^k}{k!} \langle\langle \hat{Q}_t^k \rangle\rangle.$$

In the limit $N \gg 1$, we can find $\chi(\lambda)$ exactly. The idea underlying these calculations is that the statistical averaging which appears in the definition of $\chi(\lambda)$ is interpreted as finding the vacuum expectation value of the S-matrix of an auxiliary quantum-mechanical problem.

We consider a system of $2N$ fermions with Hamiltonian (2), in the state $|0\rangle$ specified by $a_i^+ |0\rangle = 0$, $b_i |0\rangle = 0$, $(i = 1,...,N)$. We find the projection of a state of the system onto $|0\rangle$ after a time λ; i.e., we calculate the matrix element $\langle 0 | T \exp[-i \int_0^\lambda \hat{J}(\tau) d\tau] | 0 \rangle$. As usual, we expand the chronological exponential function in a series. Using Wick's theorem, we can write the expectation values as the products of T-ordered binary expectation values. It is not difficult to see that the T-ordered expectation values in terms of the state $|0\rangle$ *are exactly the same* as the statistical expectation values introduced above. On the other hand, since Hamiltonian $\hat{J}$ is the sum of N independent two-fermion Hamiltonians, the S-matrix is found as the product of N commuting two-fermion S-matrices. Accordingly, the matrix element in which we are interested is the Nth power of a two-fermion matrix element. The latter can in turn be found easily by making use of the circumstance that the Hamiltonian $\hat{J} = \Sigma_{i=1}^N \hat{J}_i$ conserves the numbers of particles: $\hat{n}_i = a_i^+ a_i + b_i^+ b_i$. We can thus rewrite each $\hat{J}_i$ in the sector of interest here $(n_i = 1)$ in terms of Pauli matrices: $\hat{J}_i = eD\hat{\sigma}_i^z + e\Lambda\hat{\sigma}_i^x$. We thus find $\chi(\lambda) = \langle \uparrow | \exp[-i\lambda e(D\hat{\sigma}^z + \Lambda\hat{\sigma}^x)] | \uparrow \rangle^N$. Evaluating the matrix element, we finally find

$$\chi(\lambda) = (cos(\lambda e\sqrt{D}) - i\sqrt{D} sin(\lambda e\sqrt{D}))^N . \tag{4}$$

One can verify that a power series of $\ln(\chi(\lambda))$ reproduces the correlation functions in (3).

We now use the characteristic function to find the distribution of charge Q_t: $P_t(Q) = \langle \delta(Q_t - Q) \rangle = \int e^{i\lambda Q} \chi(\lambda) d\lambda / 2\pi$. An unexpected property of expression (4) for $\chi(\lambda)$ is a periodicity, with a period $2\pi / e\sqrt{D}$. This periodicity leads to a quantization of Q_t. Evaluating $P_t(Q)$ as the Fourier transform of $\chi(\lambda)$, we have a discrete distribution with a quantum $2e\sqrt{D}$:

$$P_t(Q) = \sum_{m=0}^{\infty} P_m \ \delta(Q-(N-2m)e\sqrt{D}) \ ,$$

$$P_m = p^m q^{N-m} \frac{\Gamma(N+1)}{\Gamma(m+1)\Gamma(N-m+1)} \ , \tag{5}$$

where $q, p = \frac{1}{2}(1 \pm \sqrt{D})$. Here we have used a Bernoulli binomial distribution, which we have written in a form suitable for both integer and noninteger values of N. The quantity N thus takes on a probabilistic meaning: "the number of attempts."

To discuss this distribution, we naturally assume that (5) describes a classically random process, rather than a quantum-mechanical problem. We would thus like to interpret the exponential time dependence of characteristic function (4) as being a result of the *steady-state nature* of the random process.[9] This assumption naturally leads to the representation that the transition probabilities $W_{m\to m'} = dP_{m'-m}/dt_{(t=0)}$ are constant. Expressions (4) and (5) work at $t \gg \tau_V$; the correlation time of the process is thus $\ll \tau_V$.

We can offer a verbal description of the picture which arises: The amount of charge which has passed over a time t takes on one of the values $Q_m(t) = e\sqrt{D}[(2e/h)Vt - 2m]$, $(m = 0,1,...)$. At random times, there are jumps from m to $m' > m$; there is a transfer of $m' - m$ quanta $e^* = 2e\sqrt{D}$ in the direction *opposite* the direction of the average current. The time scale between these current pulses is on the order of τ_V and is much longer than a single pulse. The most probable instantaneous value of the current is $2e^2/h\sqrt{D}\,V$, not the expectation value $(2e^2/h)DV$ given by the Landauer formula (Fig. 1).

The reasons for the fluctuations are the correlations which stem from the Pauli principle. These fluctuations are not a direct result of the quantization of electric charge. The states which are participating in the charge transport are completely delocalized and are in the form of plane waves. We should thus not be surprised to find

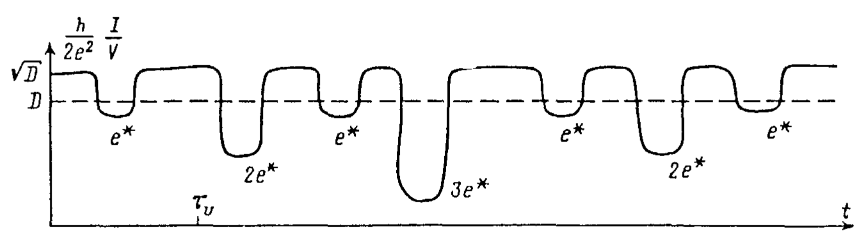

FIG. 1. Time evolution of the current corresponding to distribution (5). The average current $(2c^2/h)DV$ is lower than the most probable instantaneous value $2e^2/h\sqrt{D}\,V$. The negative pulses carry a charge which is a multiple of $e^* = 2e\sqrt{D}$.

that the quantum of charge is not an integer. On the other hand, we note that the nature of the quantized pulses of the "countercurrent" is still unclear in many ways and requires further research.

[1]Y. P. Li, D. C. Tsui, J. J. Heremans *et al.*, Appl. Phys. Lett. **57**, 774 (1990).

[2]C. Dekker, A. J. Scholten, F. Liefrink *et al.*, Phys. Rev. Lett. **66**, 2148 (1991).

[3]G. B. Lesovik, Pis'ma Zh. Eksp. Teor. Fiz. **49**, 513 (1989) [JETP Lett. **49**, 592 (1989)].

[4]B. Yurke and G. P. Kochanski, Phys. Rev. B **41**, 8184 (1990).

[5]M. Büttiker, Phys. Rev. Lett. **65**, 2901 (1990).

[6]R. Landauer, in *Localization, Interaction and Transport Phenomena,* Vol. 61 (ed. B. Kramer, G. Bergmann, and Y. Bruynseraede), Springer, Heidelberg, 1985, p. 38.

[7]A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Quantum Field-Theoretical Methods in Statistical Physics*, Pergamon, New York, 1965, § 39.

[8]A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Quantum Field-Theoretical Methods in Statistical Physics*, Pergamon, New York, 1965, § 39.

[9]I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, Nauka, Moscow, 1977.

Translated by D. Parsons