Soliton chains in a plasma with magnetic viscosity

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Two-dimensional motions in a plasma with magnetic viscosity can be described by the Kadomtsev-Petviashvili equation. Solutions of a new type—chains of screened solitons—are found for this equation.

1. We have shown previously[1,2] that in the collisionless approximation a plasma in a magnetic field should be described by the equations of anisotropic magnetohydrodynamics retaining the term corresponding to a dissipationless ion viscosity. In Ref. 3 we analyzed a solution in the form of a very nonlinear one-dimensional soliton, and we

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showed that plasma waves are excited at the soliton front in the two-fluid model. These waves grow to a wave energy $\tilde{w}$ on the order of $nT\sqrt{m_e/m_i}$. In the present letter we analyze slightly nonlinear two-dimensional waves in the single-fluid model. The equations for these waves lead to the Kadomtsev-Petviashvili equation [Eq. (4) below], for which we derive solutions of a new type, corresponding to chains of screened solitons.

2. A plasma in a magnetic field $B_z(x,y)$ is described by

$$\rho\dot{\mathbf{v}} = -\vec{\nabla}P + \eta\Delta[\mathbf{h}\mathbf{v}]; \quad \mathbf{h} = \mathbf{B}/B; \qquad P = p_\perp + B^2/8\pi, \tag{1}$$

where $\eta = p_\perp^i/2\omega_{Bi}$ is the coefficient of the collisionless ion viscosity (the "skew viscosity"[1,2,5]). The other equations,

$$\dot{\rho} = -\rho\,\mathrm{div}\,\mathbf{v}; \quad \dot{p}_\perp = -2p_\perp\mathrm{div}\mathbf{v}; \quad \dot{B} = -B\mathrm{div}\mathbf{v}, \tag{2}$$

lead to the adiabat $P = P_0(\rho/\rho_0)^2$, so that by introducing the transverse sound velocity $c_0 = \sqrt{2P_0/\rho_0}$ and the characteristic length $R_i = \eta/\rho_0 c_0$, which is on the order of the ion Larmor radius, we find

$$\frac{\partial\mathbf{v}}{\partial t} = -(\mathbf{v}\vec{\nabla})\mathbf{v} - c_0^2\vec{\nabla}\frac{\rho}{\rho_0} + c_0R_i\Delta[\mathbf{h}\mathbf{v}]; \qquad \frac{\partial\rho}{\partial t} = -\mathrm{div}\rho\mathbf{v}. \tag{3}$$

For a slight nonlinearity, this equation can be reduced to the Kadomtsev-Petviashvili equation.[4]

3. To carry out this derivation we assume that $v_{x,y}(t,x,y)$ does not depend on z, that the nonlinearity is slight, and that the derivatives with respect to y are small ($\partial/\partial y \ll \partial/\partial x$). Setting $\rho/\rho_0 = 1 + \psi$, we find from (3)

$$\psi''_{tt} - c_0^2\psi''_{xx} = c_0^2\psi''_{yy} + c_0R_i\Delta(\mathrm{rot}\,\mathbf{v})_z + \mathrm{div}(\mathbf{v}\vec{\nabla})\mathbf{v} - \mathrm{div}(\psi\mathbf{v})'_t. \tag{4}$$

All the terms on the right are assumed to be small, and they can be found by successive approximations. In the zeroth approximation we find

$$\psi'_t + c_0\psi'_x \cong 0; \quad (\mathrm{rot}_z\mathbf{v})'_t \cong c_0R_i(v_x)'''_{xxx}; \quad v_x \cong c_0\psi, \tag{5}$$

so that in the small terms we can make the replacement $\partial/\partial t \to -c_0\partial/\partial x$. Equation (4) then yields

$$c_0^{-2}\psi''_{tt} - \psi''_{xx} = \psi''_{yy} - R_i^2\psi''''_{xxxx} + \frac{3}{2}(\psi^2)''_{xx}. \tag{6}$$

We can replace the operator on the left approximately by

$$\left(\frac{\partial}{c_0\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial\psi}{c_0\partial t} + \frac{\partial\psi}{\partial x}\right) \cong -2\frac{\partial}{\partial x}\left(\frac{\partial\psi}{c_0\partial t} + \frac{\partial\psi}{\partial x}\right). \tag{7}$$

As a result, we find the Kadomtsev-Petviashvili equation,

$$(c_0^{-1}\psi'_t + \psi'_x + \frac{3}{2}\psi\psi'_x - \frac{1}{2}R_i^2\psi'''_{xxx})'_x = -\frac{1}{2}\psi''_{yy}, \tag{8}$$

which also arises in many other problems.

4. Previous study of Eq. (8) has yielded particular solutions which correspond to exponentially decaying one-dimensional solitons and also solitary two-dimensional solitons or N-soliton two-dimensional formations which fall off as r^{-2}. We will now demonstrate solutions of a new type, corresponding to horizontal and vertical chains of solitons.

For waves which are propagating along the x axis at a velocity $u < c_0$ and which do not alter the profiles, we introduce the new dimensionless arguments

$$\xi = (x - ut)/R_i; \qquad \eta = y/R_i \tag{9}$$

and the parameter $\kappa = \sqrt{1 - u/c_0}$. It is then a straightforward matter to show that the solution

$$\psi = -4\mu^2(1 + fg)(f+g)^{-2}; \quad f = \operatorname{ch}\mu\xi; \quad g = \alpha\cos q\eta \tag{10}$$

with the parameters

$$\alpha^2 = (\kappa^2 - 2\mu^2)/(\kappa^2 - \frac{1}{2}\mu^2); \quad q = \mu\sqrt{2\kappa^2 - \mu^2} \tag{11}$$

satisfies Eq. (8) and has the following properties: a) At $\mu = \kappa/\sqrt{2}$, it yields the customary one-dimensional rarefraction soliton,

$$\psi_1 = -2\kappa^2 \operatorname{ch}^{-2}(\kappa\xi/\sqrt{2}). \tag{12}$$

b) At $\mu > \kappa\sqrt{2}$, the parameter q becomes imaginary, so that we find

$$\psi_2 = -4\mu^2(1 + \alpha\operatorname{ch}X\operatorname{ch}Y)(\operatorname{ch}X + \alpha\operatorname{ch}Y)^{-2}; \quad X = \mu\xi;$$

$$Y = \eta\mu\sqrt{\mu^2 - 2\kappa^2}. \tag{13}$$

This solution corresponds to an X-shaped interaction of two oblique solitons of the original type, (12), which become distorted near the intersection region. Along the straight lines $Y = \pm X$ this solution has the behavior

$$\psi_2(X, Y = \pm X) = -\left(\frac{2\mu}{1+\alpha}\right)^2 (\alpha + \operatorname{ch}^{-2}X). \tag{14}$$

c) We can make the substitution $\alpha \to -|\alpha|$ in our original equation, (10). Then taking the limit $\mu \to 0$, we find

$$\psi_3 = -\frac{16}{3}\kappa^2\left[1 + \frac{2}{3}\kappa^2(2\kappa^2\eta^2 - \xi^2)\right]\left[1 + \frac{2}{3}\kappa^2(2\kappa^2\eta^2 + \xi^2)\right]^{-2}, \tag{15}$$

which is well known from earlier work.[5] d) The replacement $\mu \to i\mu_1$ in (10) yields a chain of solitons along the x axis:

$$\psi_4 = 4\mu_1^2(1 + |\alpha|\cos X\operatorname{ch}Y)(\cos X + |\alpha|\operatorname{ch}Y)^{-2};$$

$$X = \mu_1\xi; \quad Y = q_1\eta, \tag{16}$$

where the parameters are

$$|\alpha| = \sqrt{(\kappa^2 + 2\mu_1^2)/(\kappa^2 + \frac{1}{2}\mu_1^2)} > 1; \quad q_1 = \mu_1\sqrt{2\kappa^2 + \mu_1^2}. \tag{17}$$

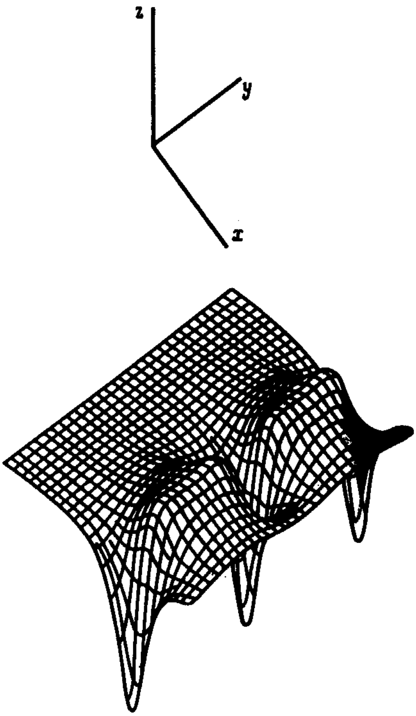

FIG. 1. Chains of solitons across the wave propagation direction (a y chain).

e) Finally, solution (10) itself with $\mu < \kappa/\sqrt{2}$ and $\alpha < 1$ leads to a chain of solitons along the y axis (Fig. 1), which may be regarded as the result of an instability of the one-dimensional soliton in (12), which was found previously[4] and which has been studied in detail in Ref. 6.

In summary, all the simplest solutions of the Kadomtsev-Petviashvili equation can be described by a common equation, (10).

In the case of a collisionless plasma with a magnetic viscosity, these solutions may be interpreted as chains of current eddies which are stretched out along the field and which are "marching" either in "column" order, one behind another,or "flank" order, parallel to each other.

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