Abelian projection of gluodynamics and confinement on a lattice

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(Submitted 5 March 1994)
Pis'ma Zh. Eksp. Teor. Fiz. **59**, No. 7, 439–441 (10 April 1994)

In lattice gluodynamics in the maximal Abelian projection, SU(2) → U(1), the dynamics is governed by diagonal gluons. In the minimal projection, it is instead governed by off-diagonal gluons. In the maximal Abelian projection, color confinement is thus due to a condensate of U(1) monopoles, while in the minimal Abelian projection the dynamics of the system is driven by vector Goldstone fields and topological defects constructed from them.

In a well-known paper, t'Hooft[1] proposed a partial fixing of the gauge in SU(N) gluodynamics, of such a nature that it did not fix the gauge group $[U(1)]^{N-1}$. Diagonal elements of the gluon field are transformed under these Abelian transformations as gauge fields, while off-diagonal elements are transformed as matter fields. Since the Abelian gauge group is compact, there are monopoles in the system. If the latter are condensed, confinement can be explained at a classical level:[2,3] A string between color charges forms as a (dual) analog of an Abrikosov string in a superconductor, with monopoles playing the role of Cooper pairs.

In this letter we show that this simple picture can be expected only in a single projection: the so-called maximal Abelian projection. In SU(2) lattice gluodynamics we present an example of an Abelian projection in which color confinement stems from topological excitations constructed from off-diagonal elements of gluon fields. In this gauge, confinement may thus be implemented not by monopoles but by "matter fields" constructed from off-diagonal gluons.

In the maximal Abelian projection[4] SU(2) → U(1) , the matrices $U_{x\mu}$ (which correspond to edges of the lattice) are put in a "maximally diagonal" form. In other words, one seeks the maximum

$$\max_{\Omega} \sum_{x,\mu} \mathrm{Tr}(U'_{x\mu}\sigma_3 U'^{+}_{x\mu}\sigma_3), \quad U'_{x\mu}=\Omega^{+}_{x} U_{x\mu} U_{x+\hat\mu}. \tag{1}$$

The summation here is over all edges of the lattice which are determined by the site x and the direction μ. The maximum is sought by varying the matrices of the gauge transformations, Ω_x, which are determined at lattice sites. The U(1) transformation

$$U_{x\mu} \to g_x U_{x\mu} g^{+}_{x+\hat\mu}, \quad (g_x)_{11}=(g_x)^{*}_{22}=e^{i\alpha_x}, \quad (g_x)_{12}=(g_x)_{21}=0 \tag{2}$$

 0021-3640/94/070459-03$10.00

leaves condition (1) invariant. Once we have parametrized the dynamical variable U by means of the standard angles, $(U_{x\mu})_{11}=\cos\phi_{x\mu}e^{i\theta_{x\mu}}$, $(U_{x\mu})_{12}=\sin\phi_{x\mu}e^{i\chi_{x\mu}}$, $U_{22}=U_{11}^*$, $U_{21}=-U_{12}^*$, we see that the angle θ transforms as a compact gauge field: $\theta_{x,x+\mu}\to\theta_{x,x+\mu}+\alpha_x-\alpha_{x+\mu}$. We also see that the angle χ transforms as the phase of a matter field of charge 2: $\chi_{x,x+\mu}\to\chi_{x,x+\mu}+\alpha_x+\alpha_{x+\mu}$. Condition (1) is rewritten as

$$\max_{\Omega}\sum_{x\mu}\cos^2(\phi'_{x\mu}). \tag{3}$$

In other words, the projection under consideration here, SU(2) → U(1), brings the angles ϕ as close as possible to zero. The plaquette action $S_{\mathcal{P}}=\frac{1}{2}\mathrm{Tr}U_1U_2U_3^+U_4^+$, expressed in terms of the angles ϕ, θ, and χ, contains several terms, each of which is invariant under U(1) gauge transformations. By virtue of condition (3), we would expect that in the MAA projection the maximum contribution to the action would come from the term proportional to the highest power of $\cos\phi_i$:

$$S_{\mathcal{P}}^A=\cos\phi_1\cos\phi_2\cos\phi_3\cos\phi_4\cos\theta_{\mathcal{P}}. \tag{4}$$

Here $\theta_{\mathcal{P}}$ is the plaquette angle: $\theta_{\mathcal{P}}=\theta_1+\theta_2-\theta_3-\theta_4$. Numerical calculations show that for values of the charge corresponding to the continuum theory $(4/g^2>2, 2)$ the $S_{\mathcal{P}}^A$ contribution is at least 0.85% of the total action, that the factors $\cos\phi_i$ fluctuate only slightly, and that the major dynamic role is played by the factor $\cos\theta_{\mathcal{P}}$, which is none other than the plaquette action of compact electrodynamics. If the terms which fix the gauge, the Faddeev–Popov determinant, and the fluctuations in $\cos\phi_i$ are all ignored, SU(2) gluodynamics therefore reduces to compact electrodynamics. This assertion explains numerous facts which show that in the MAA gauge a fundamental role is played in the dynamics of confinement by Abelian monopoles constructed from the angles $\theta_{x,x+\mu}$. For example, the "Abelian" string tension, calculated in the MAA projection in terms of the Abelian variables $\theta_{x,x+\mu}$, is almost exactly the same as the total string tension.[5] A condensate of monopoles extracted from the fields θ in the MAA projection is nonzero in the confinement region. It vanishes at a critical temperature,[6] T_c. In other projections, the condensate of monopoles does not vanish at $T=T_c$. The special role played by the MAA can also be seen from the circumstance that in it the fractal dimensionality D_f of the current lines of the monopoles is nontrivial (>1) at $T>T_c$. In other gauges, D_f is not an order parameter for a temperature-induced phase transition.[7]

We turn now to the "minimal" Abelian (MIA) gauge, in which the search for a maximum, as in (1), is replaced by a search for a minimum:

$$\min_{\Omega}\sum_{x,\mu}\mathrm{Tr}(U'_{x\mu}\sigma_3U'^+_{x\mu}\sigma_3),\quad U'_{x\mu}=\Omega_x^+U_{x\mu}\Omega_{x+\hat{\mu}}. \tag{5}$$

It is not difficult to see that in this projection, SU(2) → U(1), the maximum contribution to the plaquette action comes from a term proportional to $S_{\mathcal{P}}^I=\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4\cos\chi_{\mathcal{P}}$, where $\chi_{\mathcal{P}}=\chi_1-\chi_2+\chi_3-\chi_4$. The dynamics of the system is thus now determined by the matter fields χ_i, which interact through $\cos\chi_{\mathcal{P}}$ in a nonstandard way. The ordinary plaquette action $\cos\chi_{\mathcal{P}}$ is not invariant under U(1) gauge transformations.

Let us assume that for a given configuration of fields on the lattice the MAA gauge is fixed. Performing on the fields the gauge transformation determined by the matrices of the gauge transformations $\Omega_x = \frac{i}{2}[(-1)^{x_1+x_2+x_3+x_4}+1]\sigma_2$ (x_k are integer coordinates, and σ_2 is a Pauli matrix), we then transform to the MIA gauge. Condition (1) becomes (5), and the U(1) gauge fields θ become "vector Goldstone" fields χ (the phases of nondiagonal fields). They now play a major role in the dynamics. The monopoles which are responsible for confinement in the MAA gauge transform into specific topological defects constructed from the fields χ. We will discuss these defects in a separate publication. Here we would point out that, as in any other Abelian projection, there are U(1) monopoles in the MIA projection, but they are not important to the dynamics of the system. The major role in the dynamics of the system is played by the phase of the off-diagonal fields $\chi_{x\mu}$ and by topological defects constructed from these fields.

One of us (M.I.P.) wishes to express his deep gratitude to the Russian Basic Research Foundation (Project Code 93–02–3609) for financial support. One of us (M.N.C.) wishes to thank the Japanese institution JSPS for financial assistance within the framework of a program of support for scientists of the former USSR.

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Translated by D. Parsons