Wess-Zumino action for the orbital dynamics of ^{3}He-A

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A singularity in the dynamics of the orbital angular momentum in ^{3}He-A is explained on the basis of an anomaly which results in a transfer of angular momentum from vacuum to excitations. This process is equivalent to the creation of electron-positron pairs in an electric field. A Wess-Zumino function which is found leads to a source of angular momentum. The presence of a vortex singularity on the Fermi surface in ^{3}He-A leads to a close relationship between the orbital dynamics and the dynamics of vortices.

The vanishing of the energy gap at two points on the Fermi surface in superfluid ^{3}He-A at $\mathbf{k} = \pm k_f \mathbf{l}$ leads to several problems in the dynamics of ^{3}He-A at low temperatures. Some of them have been solved. 1) The existence of an anomalous orbital current $-(1/2)\, C_0 \mathbf{l}(\mathbf{l}\, \mathrm{curl}\, \mathbf{l})$ with $C_0 = k_F^3/3\pi^2$, which is approximately the same as the density ρ, is a consequence of a chiral anomaly which leads to an asymmetric branch of the fermion spectrum, which intersects the Fermi surface.[1,2] 2) The same branch gives rise to a nonzero state density in the presence of texture in the field of the vector $\mathbf{l}$ and thus gives rise to a nonzero density of the normal component at $T=0$ (Refs. 1–3). 3) The nonconversation of the vacuum current $\mathbf{j}$ at $T=0$ is also a consequence of a chiral anomaly: the nonconservation of the chiral current,[4]

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$$\partial_\mu J_5^\mu = \frac{e^2}{8\pi^2} F^*_{\mu\nu} F^{\mu\nu}, \qquad F^*_{\mu\nu} = \frac{1}{2} F^{\alpha\beta} e_{\mu\nu\alpha\beta} \tag{1}$$

where $F_{\mu\nu}$ is the strength of the "electromagnetic" field $\mathbf{A} = k_f \mathbf{l}$. In ^{3}He-$A$, the creation of chiral fermion excitations is accompanied by the creation of momentum from vacuum; i.e., there is a source of vacuum momentum[1,3,5]:

$$\frac{\partial \mathbf{j}}{\partial t} + \vec{\nabla} \overleftrightarrow{\pi} = k_F \mathbf{l} \frac{1}{8\pi^2} F^*_{\mu\nu} F^{\mu\nu} \cong - \frac{3}{2} C_0 \mathbf{l} \left(\operatorname{curl} \mathbf{l} \cdot \frac{\partial \mathbf{l}}{\partial t} \right). \tag{2}$$

4) The nonanalytic logarithmic behavior of the gradient energy at[6] $T=0$ stems from the null-charge phenomenon,[5] which is well known in quantum electrodynamics.[7]

In the present letter we examine the singularity in the dynamics of the orbital angular momentum. The Lagrangian formalism, when applied to the dynamics of the vector $\mathbf{l}$ (Ref. 8), contradicts an elementary analysis,[3] which leads to an additional right side in the law expressing the conservation of the internal angular momentum of Cooper pairs, $\mathbf{L} = \frac{1}{2}\rho \mathbf{l}$:

$$\frac{\partial \mathbf{L}}{\partial t} + \frac{\delta F}{\delta \vec{\theta}} = \frac{1}{2} C_0 \frac{\partial \mathbf{l}}{\partial t}, \tag{3}$$

where $\vec{\theta}$ is the orbital rotation angle of the order parameter. The projection of $\vec{\theta}$ onto $\mathbf{l}$ corresponds to a phase of a Bose condensate. Equation (3) contains both a mass conservation law and an equation for the vector $\mathbf{l}$:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \mathbf{j} = 0, \qquad \mathbf{j} = -2 \frac{\partial F}{\partial \vec{\nabla}(\vec{\theta} \cdot \mathbf{l})}; \tag{4a}$$

$$\frac{1}{2}(\rho - C_0)\left[\mathbf{l} \frac{\partial \mathbf{l}}{\partial t}\right] = \frac{\delta F}{\delta \mathbf{l}}, \qquad \delta \mathbf{l} = [\vec{\theta}, \mathbf{l}]. \tag{4b}$$

The right side of Eq. (3) may be thought of as an anomalous source of angular momentum because of the transformation of the vacuum angular momentum $\mathbf{L}$ into the angular momentum of excitation, $\mathbf{L}_{\text{exc}}$, caused by the "electric" field $\mathbf{E} = -\dot{\mathbf{A}} = -k_F \dot{\mathbf{l}}$. This transformation corresponds to the creation of electron-positron pairs in an electric field, since the role of $\mathbf{L}_{\text{exc}}$ in quantum electrodynamics is played by the charged-particle current $\mathbf{J}$. The only difference is that in ^{3}He-A the fermions should be created in any field, by virtue of the zero mass, while in real quantum electrodynamics pair creation requires a certain threshold.[9,10] If we ignore dissipation due to collisions of particles, we conclude that the current of particles should increase: $\dot{\mathbf{J}} \sim \mathbf{E}$. In ^{3}He-$A$, this effect would correspond to an increase in the angular momentum of the excitations, $\dot{\mathbf{L}}_{\text{exc}} = -(1/2) C_0 \dot{\mathbf{l}}$. The coefficient of the proportionality between $\mathbf{j}$ and $\mathbf{E}$ in ^{3}He-A is determined by the conservation of the total internal angular momentum, $(\partial/\partial t)(\mathbf{L} + \mathbf{L}_{\text{exc}}) + \delta F/\delta \vec{\theta} = 0$, while this coefficient remains unknown in quantum electrodynamics with massless fermions.

We can show that the right side of (3), like the right side of (2), is a consequence of a chiral anomaly characteristic of massless fermions. For this purpose we examine that part of the action which leads to a Schwinger source of chiral current in (1). This part of the action is unusual in that in order to write it we need to introduce an additional, fifth coordinate, as in the case of Wess-Zumino functionals[11]:

$$S_{WZ} = \kappa \frac{\hbar}{16\pi^2} \int dt d^3x dx^5 e^{\alpha\beta\gamma\mu\nu} A_\alpha F_{\beta\gamma} F_{\mu\nu} , \tag{5}$$

The integration is carried out over a five-dimensional manifold whose boundary is physical 4-space. According to the Noether theorem, in order to conserve the chiral current J^5, we would need to perform the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$ and vary the action with respect to the parameter (χ) of the transformation. A variation of S_{WZ} leads to the right side of Eq. (1).

We can express (5) in terms of the field **l**, using $A_0 = A_5 = 0$ and $\mathbf{A} = k_F \mathbf{l}$:

$$S_{WZ} = \frac{3}{2}\kappa\hbar \int d^3x \int dt dx^5 \mathbf{l}\cdot \left[\frac{\partial \mathbf{l}}{\partial t}, \frac{\partial \mathbf{l}}{\partial x^5}\right] C_0 \tag{6}$$

We can also find which conditions must be satisfied by S_{WZ} and just what the dimensionless parameter κ is. First, a variation of S_{WZ} with respect to **l** should not depend on the method used to expand the physical space. This condition is met if C_0 is independent of T (the dynamic invariance of the parameter C_0 was proved in Ref. 12). If we choose $\kappa = 1/3$, we find S_{WZ} to be that increment in the Lebedev-Khalatnikov action[8] which leads to the right side of Eq. (3), since in this case we have

$$\frac{\delta S_{WZ}}{\delta \vec{\theta}} = \left[\mathbf{l}, \frac{\delta S_{WZ}}{\delta \mathbf{l}}\right] = \frac{1}{2} C_0 \frac{\partial \mathbf{l}}{\partial t} . \tag{7}$$

Second, the change in S_{WZ} for any choice of the 5-space must be a multiple of $2\pi\hbar$. Since the difference between the two values of the action, $S^{(1)}_{WZ} - S^{(2)}_{WZ}$, for different choices of the 5-space, with the same boundary, is an integral over a closed space, while the integral of $\mathbf{l}\cdot[(\partial \mathbf{l}/\partial t),(\partial \mathbf{l}/\partial x^5)]$ over a closed surface is a topological invariant and is a multiple of 4π, we find that this second requirement holds if $3\kappa N_0$ is an integer, where $N_0 = \int d^3x C_0$. With $\kappa = 1/3$, the quantity N_0 must be an integer. That this choice of κ is correct is verified by the circumstance that in this case the internal angular momentum is quantized correctly, i.e., it is a multiple of $\hbar/2$:

$$\int d^3x (L + L_{exc}) = \frac{\hbar}{2}\int d^3x (\rho - C_0) = \frac{\hbar}{2}(N - N_0), \tag{8}$$

where N is the number of particles.

The action S_{WZ} in (5) and (6) differs from both the Wess-Zumino action introduced for ^{3}He by Valachandran[3] and the action S_{WZ} derived in Ref. 14, which depends on the momentum **k** and which vanishes when a summation is taken over momentum. The action S_{WZ} in (5) and (6) can also be written at a semiclassical level, where the dynamic variables are the particle distribution function $n(\mathbf{k},\mathbf{r})$ and the momentum-dependent phase of the order parameter, $\Phi(\mathbf{k},\mathbf{r})$, which has a vortex singularity in k-space—a boojum—at[15] $\pm k_F \mathbf{l}$. Because of the boojums, the orbital dynamics has much in common with the dynamics of vortices. If there are no boojums on the Fermi surface, as in the case of ^{3}He-B, the action would be written in terms of n and Φ as $S = S_k^{(0)} + S_p$, where the kinetic energy is $S_k^0 = (1/2)\,\Sigma_{\mathbf{k}} \int d^3x dt n\dot{\Phi}$, and S_p is the potential energy, which depends on gradients of Φ. In order to extend the dynamic equations for n and Φ to the case of a liquid with boojums, we introduce the following action functional in an expanded space:

$$S_k = \frac{1}{2}\sum_{\mathbf{k}} \int d^3x dt dx^5 \left(\frac{\partial \Phi}{\partial t}\frac{\partial n}{\partial x^5} - \frac{\partial n}{\partial t}\frac{\partial \Phi}{\partial x^5}\right) = S_k^{(0)} + S_{WZ}, \tag{9a}$$

$$S_{WZ} = \frac{1}{2}\sum_{\mathbf{k}} \int d^3x\, dt dx^5\, n\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x^5} - \frac{\partial}{\partial x^5}\frac{\partial}{\partial t}\right)\Phi\ . \tag{9b}$$

In ^{3}He-A, the difference between the mixed derivatives of the phase Φ is nonzero at the points of boojums,[15] where it is $2\pi(\mathbf{k}\cdot\mathbf{l})^2\delta(\mathbf{k}_\perp)\mathbf{l}[(\partial\mathbf{l}/\partial t),(\partial\mathbf{l}/\partial x^5)]$. As a result, after a summation over $\mathbf{k}$, expression (9b) becomes (6) with $\kappa = 1/3$.

Expression (9) can also be applied to ordinary vortices in r-space, by taking Φ in the form $\Phi(\mathbf{k}) + 2\varphi(\mathbf{r},t)$, where $\varphi(\mathbf{r},t)$ varies by $2\pi m$ when the vortex is circumvented. We then have $S_{WZ} = 2\pi\hbar m\rho(0) \int dt dx^5 d\sigma(\partial\mathbf{a}/\partial t)\cdot[(\partial\mathbf{a}/\partial x^5),(\partial\mathbf{a}/\partial\sigma)]$, where $\mathbf{a}(\sigma)$ is a displacement of the vortex line, which depends on the coordinate (σ) along the line, while $\rho(0)$ is the density on the vortex line. A variation of S_{WZ} with respect to $\mathbf{a}$ leads to an additional Magnus force acting on the vortex:

$$\frac{\delta S_{WZ}}{\delta \mathbf{a}(\sigma)} = mh\rho(0)\left[\frac{\partial \mathbf{a}}{\partial \sigma}, \frac{\partial \mathbf{a}}{\partial t}\right], \tag{10}$$

This force must be subtracted from the ordinary Magnus force, which contains the density far from the vortex, $\rho(\infty)$. The effective Magnus force thus contains the difference $\rho(\infty) - \rho(0)$. An analogous Magnus force[16] for linear objects in a ferromagnet (cylindrical domains and magnetic vortices) can also be found from a Wess-Zumino action, which would be written in the case of a ferromagnetic as $S_{WZ}^F = \int d^3x dt dx^5 M\mathbf{m}\cdot[(\partial\mathbf{m}/\partial t),(\partial\mathbf{m}/\partial x^5)]$, where $\mathbf{m}$ is the direction of the magnetic moment $\mathbf{M} = M\mathbf{m}$. Transforming to the coordinate $\mathbf{a}$ of the linear object in accordance with $\mathbf{m} = \mathbf{m}(\mathbf{r} - \mathbf{a}(t,\sigma,x^5))$, and varying with respect to $\mathbf{a}$, we find the Magnus force $Ms[(\partial\mathbf{a}/\partial\sigma),(\partial\mathbf{a}/\partial t)]$, where s is the area swept across a unit sphere by the vector $\mathbf{m}$ in the core of the linear object. For a cylindrical domain we would have $s = 4\pi$, and for an axisymmetric magnetic vortex we would have $s = 2\pi(m_z(\infty) - m_z(0))$. We do not rule out the possiblity that in vortices of superfluid ^{3}He the Magnus force depends on the topology of the core; if it does, we would have an explanation for the jump observed in the critical velocity upon a vortex transition.[17]

In ^{3}He-A, the Magnus force acting on vortices in k-space, i.e., on boojums at the points $\pm k_F\mathbf{l}$, is given by the left side of Eq. (4b). The parameter C_0 is therefore the density in the core of the vortex singularity, where there is no superfluid gap; correspondingly, we have the value $k_F^3/3\pi^2$. The quantization of the internal angular momentum in (8) corresponds to a quantization of the motion of the vortex: As a vortex ring moves between its creation and disappearance, it sweeps out a closed surface which encloses an integer number of particles[18]

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[1]R. Combescot and T. Dombre, Phys. Rev. B **33**, 79 (1986).
[2]A. V. Balatskiĭ, G. E. Volovik, and V. A. Konyshev, Zh. Eksp. Teor. Fiz. **90**, 2038 (1986) [Sov. Phys. JETP **63**, 1194 (1986)]; A. V. Balatskiĭ and V. A. Konyshev, Zh. Eksp. Teor. Fiz. (in press).
[3]G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor Fiz. **81**, 989 (1981) [Sov. Phys. JETP **54**, 524 (1981)].
[4]J. Schwinger, Phys. Rev. **82**, 664 (1951).
[5]G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 428 (1986) [JETP Lett. **43**, 551 (1986)].
[6]M. C. Cross, Low Temp. Phys. **21**, 525 (1975).
[7]L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 1177 (1954).
[8]V. V. Lebedev and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **73**, 1537 (1977) [Sov. Phys. JETP **46**, 808 (1977)].
[9]A. B. Migdal, *Fermiony i bozony v sil'nykh polyakh* (*Fermions and Bosons in Strong Fields*), Nauka, Moscow, 1978.
[10]V. N. Gribov, "Novaya gipoteza o prirode nevyletaniya kvarkov i glyuonov" ("New hypothesis regarding the nature of quark and gluon confinement"), Preprint, 1986.
[11]J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971); E. Witten, Nucl. Phys. B **223**, 422 (1983); A. P. Balachandran, V. P. Nair, and C. J. Trahern, Phys. Rev. D **27**, 1369 (1983).
[12]G. E. Volovik and A. V. Balatskii, J. Low Temp. Phys. **58**, 1 (1985).
[13]A. P. Balachandran "Solitons in ^{3}He-*B*," Preprint, Syracuse University, SU-4228-326.
[14]A. Garg, V. P. Nair, and M. Stone, "Non-Abelian bosonization and topological aspects of BCS system," Preprint NSE-ITP-86-46.
[15]G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor Fiz. **83**, 1025 (1982) [Sov. Phys. JETP **56**, 579 (1982)]; **71**, 1129 (1976) [Sov. Phys. JETP **44**, 591 (1976)].
[16]A. V. Nikiforov and É. B. Sonin, Zh. Eksp. Teor Fiz. **85**, 642 (1983) [Sov. Phys. JETP **58**, 373 (1983)].
[17]J. P. Pekola *et al.*, Phys. Rev. Lett. **53**, 584 (1984).
[18]F. D. M. Haldane and Yong-Shi Wu, Phys. Rev. Lett. **55**, 2887 (1985).

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