Analog of gravity in superfluid ^{3}He-*A*

G. E. Volovik
L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

(Submitted 11 Spetember 1986)
Pis'ma Zh. Eksp. Teor. Fiz. **44**, No. 8, 388–390 (25 October 1986)

There is a correspondence between collective boson modes in ^{3}He-*A* and gravitational waves in the general theory of relativity. The contribution of these modes to a chiral anomaly is derived.

The systematic features in the behavior of superfluid ^{3}He-*A* at low temperatures have the same origin as the anomalies and the polarization of vacuum in particle physics, since a field theory describing the interaction of chiral fermions with collective boson fields of oscillations of the order parameter $A_{\alpha i}$ has much in common with the theory of the electroweak interaction.[1] Orbital and spin-orbit waves in ^{3}He-*A* play the roles of photons and *W* bosons (Table I). In this letter we examine the interaction

TABLE I. Correspondence between the 18 collective modes in ^{3}He-*A* and bosons in particle theory. Here *Q*, the quantum number for bosons in[3] ^{3}He-*A*, plays the role of a projection of the particle's spin onto the wave vector **q**, which is chosen along **l**. Only ten modes are listed here; the other eight modes of the order parameter correspond to a generalizaton of gravitation to the case of spin-dependent tetrads $e_a^i \to e_a^{i\alpha}$ (isospin-dependent in particle physics). As a result, the number of modes associated with the gravitation is tripled (see the number of parentheses in the second column, which is the degeneracy in the weak-coupling approximation when the additional eight modes are taken into account).

Mode	Degeneracy	Oscillating variables	Variables in field theory	Analog in particle physics	*Q*
Orbital waves	2	**l**	$\mathbf{A}; g^{13}, g^{23}$	Photons	± 1
Spin-orbit waves	4	$e_{\alpha\beta\gamma} d_\beta l_i \delta A_{\gamma i}$	$\mathbf{W}_\alpha$	*W* bosons	± 1
Clapping modes	2 (6)	$(\delta\vec{\Psi}, \vec{\Delta}_1 + i\vec{\Delta}_2)$	g^{12}, $\frac{1}{2}(g^{11} - g^{22})$	Gravitons	± 2
Pseudosound	1 (3)	$\mathrm{Re}(\delta\vec{\Psi}, \vec{\Delta}_1 - i\vec{\Delta}_2)$	$\frac{1}{2}(g^{11} + g^{22})$	Additional graviton	0
Sound (+ spin waves)	1 (3)	$\mathrm{Im}(\delta\vec{\Psi}, \vec{\Delta}_1 - i\vec{\Delta}_2)$	—	Twisting wave	0

of fermion quasiparticles with the other 12 modes of the order parameter, which play the role of gravitational fields. These fields lead to an additional chiral-gravitational anomaly which effects a transfer of momentum from the superfluid vacuum to the system of quasiparticles.

To reach an understanding of the origin of the gravitional degrees of freedom, we analyze the spectrum of fermions in the field of a nonequilibrium order parameter, which, for simplicity, we assume to have fixed spin structure:

$$A_\alpha^i = d_\alpha \Psi^i . \tag{1}$$

Here $\mathbf{d}$ is the unit vector of the magnetic anisotropy in ^{3}He-A, and $\vec{\Psi}$ is an arbitrary complex vector, which at equilibrium takes the usual form for the ΦA phase:

$$\vec{\Psi}^{eg} = \Delta_0(\vec{\Delta}_1 + i\vec{\Delta}_2), \quad |\vec{\Delta}_1|^2 = |\vec{\Delta}_2|^2, \quad \vec{\Delta}_1 \cdot \vec{\Delta}_2 = 0, \tag{2}$$

where Δ_0 is the equilibrium amplitude of the gap in the excitation spectrum.

In the presence of such an order parameter is $E^2 v_F^2 (k - k_F)^2 + |\mathbf{k}\cdot\vec{\Psi}|^2/k_F^2$; this spectrum vanishes at two points on the Fermi surface: $\mathbf{k} = \pm k_F \mathbf{l}$, where $\mathbf{l} = i[\vec{\Psi},\vec{\Psi}^*]/|[\vec{\Psi},\vec{\Psi}^*]$. Expanding E^2 near these points, we find the spectrum of charged massless particles in electromagnetic and gravitational fields:

$$E^2 = g^{ij}(k_i - eA_i)(k_j - eA_j), \tag{3}$$

where $\mathbf{A} = k_F \mathbf{l}$, $e = \pm 1$, and the metric tensor $g^{\mu\nu}$ is expressed in terms of the tetrads $e_a^\mu [a = 0,1,2,3; \mu = (0,i)]$

$$g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}, \quad \eta^{11} = \eta^{22} = \eta^{33} = -\eta^{00} = 1,$$

$$e_1^i + ie_2^i = \Psi^i/k_F, \quad e_3^i = v_F l^i, \quad e_0^0 = 1, \quad e_a^0 = e_0^i = 0 \quad (a \neq 0). \tag{4}$$

The Weyl equations for these fermions can be found by "taking the square root" in (3). However, a direct derivation of these equations from the Bogolyubov equation[2] shows that the chirality of the fermions near each of the poles $\mathbf{k} = ek_F\mathbf{l}$ must be dealt with correctly. As a result, we find the following equations for left-hand negatively charged ($e = -1$) and right-hand positively charged ($e = +1$) particles:

$$\left\{ i\frac{\partial}{\partial t} - \frac{e}{2}\tau^a\left(e_a^j\left(\frac{1}{i}\partial_j - eA_j\right) + \left(\frac{1}{i}\partial_j - eA_j\right)e_a^j\right)\right\}\psi = 0, \tag{5}$$

where $\tau^a = (\tau^1,\tau^2,\tau^3)$ are the Pauli matrices.

These equations can be written in covariant form:

$$e_a^\alpha \gamma^a \nabla_\alpha \psi = 0. \tag{6}$$

This is done by introducing the connections $\omega_{\alpha,ab}$ expressed in terms of $g^{\mu\nu}$, the Christoffel symbols Γ, the twisting $A_{\gamma,\mu\nu}$, modified by twisting of the gauge field $\widetilde{A}_\alpha$, and the long derivative $\widehat{\nabla}\alpha$ (Ref. 4, for example):

$$\omega_{\alpha,\,ab} = e_a^{\nu}(\partial_\alpha e_{\nu b} - \Gamma^{\gamma}_{\alpha\nu} e_{\gamma b});$$

$$A_{\gamma,\,\mu\nu} = e_\gamma^a(\partial_\mu e_{\nu a} - \partial_\nu e_{\mu a}); \tag{7}$$

$$\tilde{A}_\alpha = A_\alpha + \frac{1}{8} g_{\alpha\beta} E^{\beta\gamma\mu\nu} A_{\gamma,\,\mu\nu},\quad E^{\beta\gamma\mu\nu} = \frac{1}{\sqrt{-g}} e^{\beta\gamma\mu\nu};$$

$$\nabla_\alpha = \partial_\alpha + \frac{1}{8}\,\omega_{\alpha,\,ab}[\gamma^a,\gamma^b] - ie\tilde{A}_\alpha,\quad \gamma^a = (e\tau^0,\tau^1,\tau^2,\tau^3).$$

The electromagnetic and gravitational variables introduced by us can be expressed in terms of the six components of the vector $\vec{\Psi}$ (Table I). Of these six, two correspond to oscillations of the vector **l**, i.e., to photons,[1] while the other four are related to gravitation. Three of these four correspond to oscillations of the metric, and one to an oscillation of the twisting tensor (this is a sound in which there is an oscillation of the phase of the condensate and thus of the angle of the rotation around **l**). The Lagrangian describing these four modes, η_n, which we choose, for simplicity, to be propagating along **l** is[3]

$$L_G = \frac{N_F}{32}\left\{\sum_{n=1}^{4}\left[\frac{1}{3}v_F^2(\partial_3\eta_n)^2 - \left(\frac{\partial}{\partial t}\eta_n\right)^2\right] + \frac{4}{3}\Delta_0^2(\eta_1^2+\eta_2^2+2\eta_3^2)\right\}, \tag{8}$$

where $\eta_1 = (\delta g^{11} - \delta g^{22}/2g^{(0)11})$, $\eta_2 = (\delta g^{12}/g^{(0)11})$, $\eta_3 = (\delta g^{11} + \delta g^{22}/2g^{(0)11})$, and η_4 corresponds to a massless acoustic wave.

The "clapping modes" in which η_1 and η_2 oscillate correspond to gravitons in the general theory of relativity[5] with a spin projection $Q = \pm 2$. The appearance of yet another graviton, η_3, with $Q = 0$, is a consequence of the noncovariance of ^{3}He-A far from poles. All the gravitons are massive. Interestingly, the massive third term in (8) might be thought of as a cosmological term if it is modifed as in Ref. 6:

$$L_{\rm cosm} = \Lambda\sqrt{-g}\left(\frac{1}{2}g^{(0)}_{\mu\nu}g^{\mu\nu} - 1\right) \cong \Lambda\sqrt{-g^{(0)}}\left\{1 + \frac{1}{2}(\eta_1^2+\eta_2^2+2\eta_3^2)\right\}, \tag{9}$$

where $\Lambda = \Delta_0^4/12\pi^2$, and $g^{(0)\mu\nu}$ is the equilibrium metric tensor corresponding to Minkowski space: $[g^{(0)00} = -1.\ g^{(0)ij} = c_\parallel^2 l^i l^j + c_\perp^2(\delta^{ij} - l^i l^j),\ c_\parallel = v_F,\ c_\perp = \Delta_0/k_F$. In the derivation of the second equation in (9) we imposed the condition $l_i\delta g^{ij} = 0$ in order to eliminate the nonphysical variable δg^{33} and those variables (δg^{13} and δg^{23}) which correspond to oscillations of the vector **l**, which are already incorporated in the "electromagnetic" waves. Since we are dealing with wave vector $\mathbf{q}\|\mathbf{l}$, this condition is equivalent to the condition $\partial_\mu g^{\mu\nu} = 0$.

The interaction of electromagnetic and gravitational fields with massless fermions, which is described by Eq. (6), should give rise to an anomaly in the chiral current.[7] This anomaly would result from not only the "electromagnetic" field $\mathbf{A} = k_F\mathbf{l}$ but also the "grativation," i.e., the Riemannian curvature R and the twisting which enters the effective field $\tilde{A}_\alpha$:

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2} e^{\mu\nu\sigma\tau}\left(\tilde{F}_{\mu\nu}\tilde{F}_{\sigma\tau} + \frac{1}{24}R^{\alpha}_{\ \beta\mu\nu}R^{\beta}_{\ \alpha\sigma\tau}\right), \tag{10}$$

where $\widetilde{F}_{\mu\nu} = \partial_\mu \widetilde{A}_\nu - \partial_\nu \widetilde{A}_\mu$. This circumstance leads to an additional exchange of momentum between the superfluid vacuum and excitations.[1] The effective $\widetilde{A}_0$ is $k_F \mathbf{l} \cdot \mathbf{v}_s + (1/4) \mathbf{v}_F \mathbf{l}\, \text{curl}\, \mathbf{l}$, in contrast with the value $k_F \mathbf{l} \cdot \mathbf{v}_s$ cited in Ref. 1.

[1]G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 428 (1986) [JETP Lett. **43**, 551 (1986)]; **44**, 144 (1986) [JETP Lett. **44**, 185 (1986)].

[2]A. V. Balatskiĭ, G. E. Volovik, and V. A. Konyshev, Zh. Eksp. Teor. Fiz. **90**, 2038 (1986) [Sov. Phys. JETP (to be published)].

[3]P. N. Brusov and V. N. Popov, Zh. Eksp. Teor. Fiz. **78**, 234 (1980) [Sov. Phys. JETP **51**, 117 (1980)]; **79** 1871 (1980) [Sov. Phys. JETP **52**, 945 (1980)].

[4]L. D. Faddeev, Usp. Fiz. Nauk **136**, 435 (1982) [Sov. Phys. Usp. **25**, 130 (1982)].

[5]L. D. Landau and E. M. Lifshitz, Teoriya polya, Moscow, 1967 (The Classical Theory of Fields), Addison-Wesley, Reading, Mass., 1971.

[6]A. A. Logunov and Yu. M. Loskutov, Teor. Mat. Fiz. **67**, 323 (1986).

[7]L. Alvarez-Gaume and P. Ginsparg, Ann. Phys. **161**, 423 (1985).

[8]G. E. Volovik, and M. V. Khazan, Zh. Eksp. Teor. Fiz. **85**, 948 (1983) [Sov. Phys. JETP **58**, 551 (1983)].

Translated by Dave Parsons