Lagrange equations of the hydrodynamics of the anisotropic superfluid liquid He^3-*A*

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The Lagrangian formalism is used to derive equations that describe the hydrodynamics of He^3-*A*. The form of the conservation laws is obtained. The dissipative terms in the hydrodynamic equations are considered.

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A Lagrangian formalism for the description of the hydrodynamics of He–II was developed in[1]. This method can be generalized to include the case of the anisotropic superfluid liquid He^3–A. Compared with He–II, this phase has additionally an order parameter $\Phi = \Phi_1 + i\Phi_2$ (we disregard the spin variables), and the energy ϵ acquires a dependence on $\mathbf{l}$ and on $\nabla i\mathbf{l}$ ($\mathbf{l}$ is the angular momentum of the pair). We add to the Lagrangian of[1] a term that describes the transport Φ_1 (in which case Φ_2 appears automatically as a Lagrangian multiplier), as well as the structural condition on the reference frame Φ_1, Φ_2, $\mathbf{l}$. As a result we get

$$L = \tfrac{1}{2}\rho v_s^2 - \mathbf{j}\mathbf{v}_s + \tilde{\epsilon}(\rho, s, \mathbf{v}_s - \mathbf{v}_n, \mathbf{l}, \nabla_i l_k) - a\,(\dot{\rho} + \nabla\mathbf{j}) - \beta(\dot{s} + \nabla(s\mathbf{v}_n))$$
$$- \gamma(\dot{f} + \nabla(f\mathbf{v}_n)) - \vec{\Phi}_2 \frac{d}{d\tau}\vec{\Phi}_1 - g(1 - [\vec{\Phi}_1, \vec{\Phi}_2]), \tag{1}$$

where $d/d\tau = \hbar/2m[\rho(\partial/\partial t) + \mathbf{j}\nabla]$, f, γ are the Clebsch variables, $\tilde{\epsilon} = \epsilon - \mathbf{p}(\mathbf{v}_n - \mathbf{v}_s)$, $\mathbf{p} = \partial\tilde{\epsilon}/\partial\mathbf{v}_s$ is the normal momentum density.[1)] The Lagrangian (1) leads to the system of equations

$$\dot{\rho} + \nabla\mathbf{j} = 0 \qquad \dot{s} + \nabla(s\mathbf{v}_n) = 0 \qquad \dot{f} + \Delta(f\mathbf{v}_n) = 0, \tag{2}$$

$$\mathbf{l} = [\vec{\Phi}_1 \times \vec{\Phi}_2] \qquad \mathbf{v}_s = \nabla a - \frac{\hbar}{2m}\Phi_{2i}\nabla\Phi_{1i}, \tag{3}$$

$$\dot{\beta} + \mathbf{v}_n\nabla\beta + T = 0 \qquad \dot{\gamma} + \mathbf{v}_n\nabla\gamma = 0 \qquad \dot{a} + \tfrac{1}{2}v_s^2 + \mu - \frac{\hbar}{2m}\Phi_2\dot{\Phi}_1 = 0, \tag{4}$$

$$\mathbf{p} = s\,\nabla\beta + f\,\nabla\gamma \qquad \mathbf{j} = \mathbf{p} + \rho\,\mathbf{v}_s, \tag{5}$$

$$\frac{d}{d\tau}\vec{\Phi}_a = [\,\mathbf{g}\times\vec{\Phi}_a\,] \qquad g_i = \frac{\partial\epsilon}{\partial l_i} - \nabla_k \frac{\partial\epsilon}{\partial(\nabla_k l_i)}. \tag{6}$$

We note that the term $\nabla\alpha$ in expression (3) for the superfluid velocity can always be eliminated by a gauge transformation of the order parameter $\mathbf{\Phi}$. From (6) it follows that $d(\mathbf{\Phi}_\alpha\cdot\mathbf{\Phi}_\beta)/d\tau = 0$, i.e., the relations $\mathbf{\Phi}_\alpha\mathbf{\Phi}_\beta = \delta_{\alpha\beta}$ can be regarded as boundary conditions for the given system of equations. Taking these relations into account, we can obtain from (3) an expression for the curl of the superfluid velocity

$$\nabla_i v_{sj} - \nabla_j v_{si} = \frac{\hbar}{2m}\,\mathbf{l}[\nabla_i\mathbf{l}\times\nabla_j\mathbf{l}\,]. \tag{7}$$

It follows directly from (3) and (6) that

$$d\mathbf{l}/d\tau = [\,\mathbf{g}\times\mathbf{l}\,]. \tag{8}$$

For the superfluid velocity we can obtain

$$\frac{\partial v_{si}}{\partial t} = -\nabla_i\,(\mu + \tfrac{1}{2}v_s^2) + \frac{\hbar}{2m}\,\dot{\mathbf{l}}[\nabla_i\mathbf{l}\times\mathbf{l}]. \tag{9}$$

The obtained system of equation leads to an energy conservation law

$$E = \tfrac{1}{2}\rho\,v_s^2 + \mathbf{p}\,\mathbf{v}_s + \epsilon:$$

$$\dot{E} + \nabla\mathbf{Q} = 0,$$

$$\mathbf{Q} = \mathbf{j}\,(\mu + v_s^2/2) + s\mathbf{v}_n T + (\mathbf{p}\,\mathbf{v}_n)\mathbf{v}_n - \dot{l}_i\,\frac{\partial\epsilon}{\partial(\nabla l_i)} \tag{10}$$

We note that the invariance of E to rotations leads to the identity

$$\left[\mathbf{l}\times\frac{\partial\epsilon}{\partial\mathbf{l}}\right] + \left[\nabla l_k\times\frac{\partial\epsilon}{\partial\nabla l_k}\right] + \left[\nabla_k\mathbf{l}\times\frac{\partial\epsilon}{\partial\nabla_k\mathbf{l}}\right] + [\,\mathbf{p}\times(\mathbf{v}_n - \mathbf{v}_s)] = 0. \tag{11}$$

From the Lagrangian (1) can obtain in standard fashion the stress tensor, which we shall use to formulate the momentum conservation law

$$\frac{\partial j_i}{\partial t} + \nabla_k\pi_{ik} = 0,$$

$$\pi_{ik} = \delta_{ik}P + \nabla_i\mathbf{l}\frac{\partial\epsilon}{\partial\nabla_k\mathbf{l}} + j_k v_{si} + v_{nk}p_i, \tag{12}$$

where $P = Ts + \mu\rho + \mathbf{p}(\mathbf{v}_n - \mathbf{v}_s) - \epsilon$ is the pressure. We note that the tensor π has an antisymmetrical part that can be reduced with the aid of the identity (11) to the form

$$e_{nik}\pi_{ik} = \nabla \mathbf{B}_n + \frac{\hbar}{2m}\frac{\partial}{\partial t}(\rho l_n), \tag{13}$$

where

$$\mathbf{B}_n = \frac{\hbar}{2m}\mathbf{l}\, j_n - \left[\mathbf{l}\times\frac{\partial\epsilon}{\partial\nabla_n \mathbf{l}}\right].$$

The angular momentum conservation law takes thus the form

$$\frac{\partial}{\partial t}\left([\mathbf{r}\times\mathbf{j}] + \frac{\hbar}{2m}\rho\mathbf{l}\right) = -\nabla_k([\mathbf{r}\times\vec{\pi}_k] + \mathbf{B}_k). \tag{14}$$

We proceed now to consider the dissipative terms. By virtue of the condition $l^2=1$ the generalization of Eq. (8) is effected by the substitution $\mathbf{g}\to\mathbf{g}+(\hbar\rho/2m^{\mathbf{u}})$, and the generalization of (9) is realized, by virtue of the condition (7), by putting $\mu\to\mu+h$. We add to π_{ik} the dissipative part $\tau_{ik}+\lambda\delta_{ik}$ $(\tau_{ii}=0)$, we rewrite the equation for the entropy growth in the form $T(\dot{s}+\nabla(s\mathbf{v}_n+\mathbf{q}/T))=R$, and obtain by using the standard procedure[2] $E+\nabla(\mathbf{Q}+\mathbf{Q}')=0$, where

$$Q' = \mathbf{j}_0 h + \lambda\mathbf{v}_n + v_{ni}\vec{\tau}_i + \mathbf{q} - \frac{\partial\epsilon}{\partial(\nabla l_i)}[\mathbf{u}\times\mathbf{l}]_i, \tag{15}$$

$$R = -h\nabla\mathbf{j}_0 - \lambda\nabla\mathbf{v}_n - \frac{\mathbf{q}}{T}\nabla T - \mathbf{u}[\mathbf{l}\times\mathbf{g}] - \tau_{ik}w_{ik}, \tag{16}$$

and we have introduced $\mathbf{j}_0=\mathbf{j}-\rho\mathbf{v}_n$, $\omega_{ik}=\frac{1}{2}(\nabla_i v_{nk}+\nabla_k v_{ni})-\frac{1}{3}\delta_{ik}\nabla\mathbf{v}_n$. In the approximation linear in the spatial derivatives we obtain

$$h = -\zeta_1\nabla\mathbf{j}_0 - \zeta_2\nabla\mathbf{v}_n - \zeta_3(\mathbf{l}w\mathbf{l}),$$

$$\lambda = -\phi_1\nabla\mathbf{j}_0 - \phi_2\nabla\mathbf{v}_n - \phi_3(\mathbf{l}\,w\mathbf{l}),$$

$$\frac{\mathbf{q}}{T} = -\kappa_1(\nabla T - \mathbf{l}(\mathbf{l}\nabla T)) - \kappa_2[\mathbf{l}\times\nabla T] - \kappa_3\mathbf{l}(\mathbf{l}\nabla T), \tag{17}$$

$$\mathbf{u} = -\xi_1(w\mathbf{l}) - \xi_2[\mathbf{l}\times(w\mathbf{l})] - \xi_3\mathbf{g} - \xi_4[\mathbf{l}\times\mathbf{g}],$$

$$\tau_{ik} = -\eta_1\nabla\mathbf{j}_0 l_i l_k - \eta_2\nabla\mathbf{v}_n l_i l_k - \eta_3(\mathbf{l}w\mathbf{l})l_i l_k - \eta_4 e_{imn}l_m w_{nk}$$

$$-\eta_5 l_k((wl)_i - l_i(\mathbf{l}w\mathbf{l})) - \eta_6 l_k[l,(wl)]_i - \eta_7(g_i - l_i(\mathbf{l}\mathbf{g}))$$

$$-\eta_8 l_k[l,g]_i - \eta_9(w_{ik} - l_i(wl)_k - l_k(wl)_i + l_i l_k(\mathbf{l}w\mathbf{l})).$$

Here $\zeta,\phi,\kappa,\xi,\eta$ are the kinetic coefficients, and τ_{ik} is obtained from $\tilde{\tau}_{ik}$ by symmetrization and separation of the trace. The Onsager-symmetry requirements leads to the necessary symmetry of the matrices

$$\begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \\ \phi_1 & \phi_2 & \phi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix} \quad \begin{pmatrix} \eta_5 & \eta_7 \\ \xi_2 & \xi_4 \end{pmatrix} \quad \begin{pmatrix} \eta_5 & \eta_8 \\ -\xi_1 & \xi_4 \end{pmatrix} \tag{18}$$

By virtue of the positive-definiteness, the quadratic forms constructed from the matrices (18) should be positive-definite; in addition, we must have $\kappa_1 > 0$, $\kappa_3 > 0$, $\eta_9 > 0$, while κ_2, η_4, η_6, and ξ_3 are arbitrary.

The obtained system of hydrodynamic equations agrees in the main with Ho's system,[3] but there are some difference, due to the absence of terms with curl $\mathbf{v}_n$. In addition, in our equation $\mathbf{l}$ is transported with a velocity $\mathbf{j}/\rho$, but this quantity can be redefined by redesignating the kinetic coefficients.

It is seen from that by using the redefinition $\mathbf{j}' = \mathbf{j} + (1/2)\mathrm{curl}(\hbar\rho\mathbf{l}/2m)$ we can recast the conservation laws for the momentum and angular-momentum densities in the usual form with a symmetrical stress tensor. Equation (8) still retains its form if we use the change of notation

$$\mathbf{j} \to \mathbf{j}', \quad \mathbf{g} \to \mathbf{g} + \tfrac{1}{2}(\rho \,\mathrm{rot}\, \mathbf{v}_s + \frac{\hbar}{2m}[\,l, ([\nabla\rho \times \mathbf{l}]\nabla)\, l\,]).$$

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[1)]The considered Lagrangian formalism can be reduced to a Hamiltonian formalism by the standard procedure.

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