"Irreversibility" of the flux of the renormalization group in a 2D field theory

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There exists a function $c(g)$ of the coupling constant g in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where c is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

The renormalization group is one of the most powerful methods for qualitative studies in field theory.[1,2] The procedure for determining the renormalization group can be roughly summarized as follows[1]: We denote by $S(g,a)$ an action functional of a (Euclidean) field theory which is an integral of the local density, $S=\int\sigma(g,a,x)dx$, equipped with an ultraviolet cutoff a and depending on a (possibly infinite) set of dimensionless parameters $g=(g^1,g^2,\ldots)$, which are known as "coupling constants." A basic assumption is that there exists a single-parameter group of motions in the space (Q) of coupling constants g, $R_t\,Q\to Q$, of such a nature that a field theory describable by an action $S(R_t g,\, e^t a)$ is equivalent to the original theory with the action $s(g,a)$ in the sense that all the correlation functions of the two theories agree at scales $x \gg e^t a\,(t>0)$. The components of the vector field which generate the renormalization group are called "β functions":

$$dg^i = \beta^i(g)dt. \tag{1}$$

Some of the information on the ultraviolet behavior of the field theory is lost under renormalization transformations with $t>0$, since in the field theory it is not legitimate to examine correlations at scales smaller than the cutoff. We would therefore expect that a motion of the space Q under the influence of the renormalization group would become an "irreversible" process, similar to the time evolution of dissipative systems. In the present letter we restrict the discussion to a 2D field theory, and for this case we establish the following general properties of the renormalization group:

1. There exists a function $c(g)\geqslant 0$ of such a nature that we have

$$\frac{d}{dt}c \equiv \beta^i(g)\frac{\partial}{\partial g^i}\,c(g) \leqslant 0 \tag{2}$$

(a repeated index implies a summation). The equality in (2) is reached only at fixed points of the renormalization group, i.e., at $g=g_*$ $[\beta^i(g_*)=0]$.

2. The fixed points (here and below, we mean "critical" fixed points, at which the correlation radius is infinite[1]) are stationary for $c(g)$; i.e., we have $\beta^i(g)=0\to\partial c/\partial g^i=0$. At the critical fixed points, the 2D field theory has an infinite conformal

symmetry.[3] The corresponding generators L_n, $n=0, \pm 1, \pm 2, \ldots$, form a Virasora algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{c}}{12}(n^3-n)\delta_{n+m,0} \quad , \tag{3}$$

where the numerical parameter $\tilde{c}$ (the "central charge") is an important characteristic of a conformal field theory.[3,4] It generally takes on different values for different fixed points; i.e., $\tilde{c} = \tilde{c}(g_*)$.

3. The value of $c(g)$ at the fixed point g_* is the same as the corresponding central charge in (3); i.e., $c(g_*) = \tilde{c}(g_*)$.

The proof of this "c-theorem" is based on the conditions of renormalizability, positivity,[5] the translational and rotational symmetries of the field theory, and certain special properties of a 2D conformal field theory. Spatial symmetries in a local field theory lead to the existence of a local energy-momentum tensor $T_{\mu\nu}(x) = T_{\nu\mu}(x)$ which satisfies the equation $\partial_\mu T_{\mu\nu} = 0$. We introduce the complex coordinates $(z,z) = (x^1 + ix^2, x^1 - ix^2)$, and we use the notation $T = T_{zz}$, and $\Theta = T_{z\bar{z}}$. We also define the scalar local fields

$$\Phi_i(g,x) = \frac{\partial}{\partial g^i}\sigma(g,a,x). \tag{4}$$

The exact meaning of the assertion that a field theory is renormalizable is that for all g the field Θ can be expanded in basis (4):

$$\Theta = \beta^i(g)\Phi_i \ , \tag{5}$$

where the coefficients $\beta^i(g)$ are the same as in (1). We define the functions

$$C(g) = 2z^4 \langle T(x)T(0)\rangle |_{x^2 = x_0^2} \ ; \tag{6a}$$

$$H_i(g) = z^2x^2 \langle T(x)\Phi_i(0)\rangle |_{x^2 = x_0^2} \ ; \tag{6b}$$

$$G_{ij}(g) = x^4 \langle \Phi_i(x)\Phi_j(0)\rangle |_{x^2 = x_0^2} \tag{6c}$$

where $x_0 \gg a$ is an arbitrary scale ("normalization point"). At this point we set $x_0 = 1$. By virtue of the positivity condition in the field theory,[5] the symmetric matrix $G_{ij}(g)$ is positive definite and may be thought of as the metric in Q. Combining the requirement $\partial_\mu T_{\mu\nu} = 0$ with (5) and with the Callan-Simanchik equation,[2] we find the relations

$$\frac{1}{2}\beta^i\partial_i C = -3\beta^i H_i + \beta^i\beta^k\partial_k H_i + \beta^k(\partial_k\beta^i)H_i \ ; \tag{7a}$$

$$\beta^k\partial_k H_i + (\partial_i\beta^k)H_k - H_i = -2\beta^k G_{ik} + \beta^j\beta^k\partial_k G_{ij} + \beta^j(\partial_i\beta^k)G_{jk} + \beta^j(\partial_j\beta^k)G_{ik} \ , \tag{7b}$$

where $\partial_i = \partial/\partial g^i$. In deriving (7) we made use of the following expression for the matrix $\gamma_i^j(g)$:

$$\gamma_i^j(g)\,\Phi_j \equiv \left(\frac{1}{2}\,a\frac{\partial}{\partial a} - \beta^k\frac{\partial}{\partial g^k}\right)\Phi_i = (\partial_i\beta^j)\Phi_j \tag{8}$$

For the function

$$c(g) = C(g) + 4\beta^i H_i - 6\beta^i\beta^j G_{ij} \tag{9}$$

we find from (7)

$$\beta^i\,\partial_i c = -\,12\beta^i\beta^j G_{ij}\,, \tag{10}$$

directly verifying Assertion 1. Assertion 3 follows from (9) and from the definition of the central charge $\tilde{c}(g_*)$ as the numerical coefficient in the correlation function[3] $\langle T(z)T(0)\rangle_{g_*} = z^{-4}\tilde{c}(g_*)/2$. To prove Assertion 2, we consider the critical fixed point g_*, and we choose a coordinate system in Q such that we have $g_* = 0$ and

$$G_{ij}(g) = \delta_{ij} + 0(g^2)\,. \tag{11}$$

In this case the vectors $\Phi_i(g_*,x)$ are conformal fields and have certain anomalous dimensionalities d_i. Near the point $g_* = 0$, the function $c(g)$ can be calculated by perturbation theory; the result is

$$c(g) = \tilde{c}(g_*) - 6\epsilon_i g^i g^i + 2C_{ijk}\,g^i g^j g^k + O(g^4), \tag{12}$$

where $2\varepsilon_i = 2 - d_i$. Assertion 2, in particular, follows from (12). We also note that in the special case of "soft" perturbations, with $|\epsilon_i| \ll 1$, it can be shown that the coefficients C_{ijk} in (12) are the same as the structure constants of the operator algebra of a conformal field theory,[3] g_*. For the β functions we find

$$\beta^i(g) = \epsilon_i g^i - \frac{1}{2}C_{ijk}\,g^j g^k + O(g^3)\,. \tag{13}$$

(No summation is to be carried out in the first term.) At the specified accuracy, therefore, the following relation holds near the fixed point:

$$\beta^i(g) = -\,\frac{1}{12}\,G^{ij}(g)\,\frac{\partial}{\partial g^j}c(g)\,, \tag{14}$$

where $G^{ik}G_{kj} = \delta^i_j$.

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