Modulation instability and stong turbulence in media with inverse dispersion

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The modulation instability of an oscillation with frequency in the form $\omega = \omega_0(n) - \alpha\omega_0/2k^2$ can lead to the onset of solitons whose characteristic feature is the presence of breaks or discontinuities of the electric potential; as a result, the solitons are attenuated by thermal particles.

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The frequency $\omega = \omega_0(n) - \alpha\omega_0/2k^2$ is a characteristic of ion Langmuir oscillations $[\omega_0^2 = 4\pi n e^2/M,\ \alpha = r_D^{-2} \equiv 4\pi n e^2/T_e,\ 1 \ll k^2 r_D^2 \ll (T_e/T_i)^{1/2}]$, of short-wave electron Langmuir oscillations in a plasma to which $n' \ll n$ hot electrons have been added $[\omega_0^2 = 4\pi n e^2/m,\ \alpha = n'T/2\pi r_D^2 T',\ (n'T'/nT)^{1/2} \gg k^2 r_D^2 (T'/T) \gg 1]$, and of flute oscillations in a magnetic field near the frequency $\omega_0^2 = \omega_{pi}^2(1 + \omega_{pe}^2/\omega_{He}^2)^{-1}$ $[\alpha = -(M/m)k_z^2,\ \omega_{He}/v_{Te} \gg k \gg k_z\sqrt{M/m},\ \omega_{pe}/c]$.

These oscillations can be subject to modulation instability. Its nature can be easily understood from the following arguments. Let a wide packet of waves with amplitude E and with wave vector k pass through a region with $\delta n < 0$. The group velocity of the waves decreases in this case $\delta(\partial\omega/\partial k) = -3\alpha\omega_0 k^{-4}\delta k = (\partial\omega/\partial n)(\delta n/k)$, since $\delta\omega = (\partial\omega_0/\partial n)\delta n + (\alpha\omega_0/k^3)\delta k = 0$. Consequently δE^2

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$= -E^2\delta(\partial\omega/\partial k)/(\partial\omega/\partial k) = -E^2(3/\alpha)(\partial\ln\omega_0/\partial n)k^2\delta n$. The action of the excess pressure of the RF field, $-\nabla|E|^2/4\pi$, leads to a further decrease of the density in accordance with the equation

$$\frac{\partial^2}{\partial t^2}\delta n - C_s^2\frac{\partial^2}{\partial x^2}\delta n = \frac{\partial^2}{\partial x^2}\frac{|E|^2}{4\pi M}. \tag{1}$$

From the solution of the equation for the complex amplitude E, averaged over the frequency ω_0,[1]

$$\frac{\partial^2}{\partial x^2}\left(2i\frac{\partial E}{\omega_0\partial t} - 2\frac{\partial\omega_0}{\partial n}\delta n E\right) = aE. \tag{2}$$

it is easy to obtain an expression for the instability growth rate γ of a periodic wave modulated by a perturbation with a wave vector q. Thus, in the limit $q \ll k$, $\gamma^2 \ll q^2C_s^2$ we have

$$\left(\frac{\gamma}{\omega_0}\right)^2 = q^{-4}\left[\frac{E^4k^4}{(\pi nT)^2}\frac{d\ln\omega_0}{d\ln n} - a^2\right]. \tag{3}$$

Just as in the case of an ordinary Langmuir turbulence ($n'T' \ll nT$) there exist solutions of Eqs. (1) and (2), $E(x,t) = \exp(iqx - i\Omega t)E(x - at)$, which describe stationary modulated waves and, as a limit, solitons.[2]

For the case $q \ll (\partial/\partial x)\ln E(x - at)$, the solitons are described by the equation

$$\left(e^2 - \frac{1}{2}\right)^2\left(\frac{de}{d\xi}\right)^2 = e^2(1 - e^2), \qquad e^2 = \frac{3E^2}{8\pi T\Omega}\frac{d\omega_0}{dn}\left(\frac{a^2}{C_s^2} - 1\right)^{-1},$$
$$\xi^2 = \frac{(x - at)^2 a\omega_0}{8\Omega}. \tag{4}$$

A characteristic feature of the solutions of Eq. (4) is $de/d\xi \sim (\xi - \xi_0)^{-1/2}$ at $e^2 \to \frac{1}{2}$. In this region Eq. (2) must be supplemented by terms with higher-order derivatives in the left-hand side, of the type $-(T/m\omega_0^2)(\partial^4E/\partial x^4)$, which limit the value of $\partial E/\partial x$. The presence of a large derivative leads to damping of the oscillations by the thermal particles. The short-wave component in the Fourier expansion of the spectrum of the solution of Eq. (4) can be estimated at $E_k \approx E_0k_0^{1/2}/k^{3/2}$, where E_0 and k_0^{-1} are the amplitude and the modulation width, which are connected by the relation

$$E_0^2k_0^2 = \frac{\pi}{3}nTa\left(\frac{a^2}{C_s^2} - 1\right)\left(\frac{d\ln\omega_0}{d\ln n}\right)^{-1}. \tag{5}$$

Substituting the value of E_k in the quasilinear equation for the thermal electrons and ions, we obtain

$$\frac{\partial f}{\partial t} = N\pi\frac{e^2}{m^2}\frac{\partial}{\partial v}\int|E_k|^2\delta(\omega_0 - kv)dk\frac{\partial f}{\partial v}, \tag{6}$$

from which it follows that (N is the density of the modulations and of the solitons)

$$\frac{\partial}{\partial t}nT = -\frac{\partial}{\partial t}\frac{E_0^2}{8\pi}\frac{N}{k_0} = \frac{N}{k_0}\frac{E_0^2}{8\pi}\omega_0\frac{k_0^2v_T^2}{\omega_0^2}. \tag{7}$$

Solutions with $q \ll k_0$, described by Eqs. (4)—(7), exist if

$$a\left(\frac{a^2}{C_s^2} - 1\right)\frac{\partial\ln\omega_0}{\partial\ln n} > 0.$$

This condition is satisfies for the electron branch at $a > C_s$, for ion flute oscillations in a magnetic field at

$$a < C_s \left(\frac{\partial \ln \omega_0}{\partial n} = \frac{1}{2}\omega_H^2(\omega_p^2 + \omega_H^2)^{-1}\right)$$

and is not satisfied for short-wave ion Langmuir oscillations.

Equations (1) and (2) have also solutions in the form of wave packets $q > (\partial/\partial x) \ln E(x - at)$. They assume an interesting form in the limit when

$$\Omega = -\frac{9a}{16q^2}\omega_0 (1 + \epsilon) + \frac{1}{4} qa, \quad \epsilon << 1,$$

$$(g^2 - 1)^2\left(\frac{dg}{dx}\right)^2 = \frac{q^2}{9} g^2[(g^2 - 1)^2 - \epsilon^2],$$

$$g^2 = \frac{E^2}{2\pi T}\frac{\partial \ln \omega_0}{\partial n}\frac{8q}{3\left(a - \frac{4q^3 a}{\omega_0}\right)}\left(1 - \frac{a^2}{C_s^2}\right)^{-1} \tag{8}$$

$$g = \begin{cases} \exp\left(-\frac{q}{3}|x - at|\right) & 1 - g^2 >> \epsilon \\ 1 - \left[\frac{q^2}{9}(x - at)^2 + \frac{\epsilon^2}{4}\right]^{1/2}, & 1 - g^2 << 1. \end{cases} \tag{9}$$

This solution, in contrast to (4), has only one discontinuity of the derivative in the region $qx \lesssim 3$, but contains also short-wave ($k < q/3$) components in the Fourier expansion

$$E_k = \frac{q}{3} E_0 \left[\left(\frac{q}{3}\right)^2 + (k - q)^2\right]^{-1},$$

which leads to damping of the solution on thermal particles in accordance with the law [cf. (6), (7)].

$$\frac{\partial}{\partial t} nT = -\frac{\partial}{\partial t}\frac{E_0^2}{8\pi}\frac{3N}{q} = \frac{3N}{q}\omega_0\left(\frac{qv_T}{\omega_0}\right)^3\frac{E_0^2}{8\pi}. \tag{10}$$

Formulas (8)—(10) describe the behavior of shortwave ion Langmuir oscillations and of subsonic packets of electronic Langmuir oscillations in the presence of hot electrons.

In contrast to the Langmuir solitons, the solitons described by Eqs. (4) and (8) cannot collapse,[1] since their amplitude is proportional to the width.

Let us list certain phenomena when the considered modulations of the oscillations and of the plasma density can change place:

1) Runaway electrons in tokamaks can build up modified electron Langmuir waves and heat the bulk of the electrons.

2) Strong-current electron beams can heat the plasma during the last stage of quasilinear relaxation if $n_b m v_b^2 > nT$.

3) Formation of an energy component upon excitation of a Langmuir turbulence by beams of electrons and by light and when the condition $n'T' > nT$ is reached can cause a change in the character of the turbulence.

4) When a strong ion-sound turbulence is excited by a current, the bulk of the ion mass can become heated. Let us show how to estimate the steady-state current velocity u in the case of turbulent heating. From the momentum balance for the ions it follows that

$$enE_{\text{ext}} = M\frac{d}{dt}\int \mathrm{v}\, f_i\, dv = 3N\frac{E_o^2}{8\pi}q\frac{q^2 v_{Ti}^2}{\omega_{pi}^2} = \frac{d}{dt}nT_i\frac{1}{v_{Ti}} .$$

Since $dT_e/dT = euE_{\text{ext}}$, it follows that $T_i/T_e \approx v_{Ti}/u$ or $T_e/T_i \approx u^2/C_s^2$. The damping of the solitons by the ions should be balanced by their buildup by the current. Hence $u/v_{Te} = (T_i/T_e)^{3/2}(qr_D)^5$. Combining these relations we obtain

$$\frac{u}{v_{Te}} \approx \left(\frac{m}{M}\right)^{1/3} qr_D .$$

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