Supersymmetrical model of some classical particles with spin

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A system of classical particles with spin, to which strong coupling is applied, is examined within the context of supersymmetrical formalism. The model is a supersymmetrical analog of a classical solid. A structure Hamiltonian and Euler equations are obtained.

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The nonrelativistic dynamics of particles with spin was investigated[1] by introducing the Grassman variables for the spin degrees of freedom. The equations of motion were derived from the action principle with a quadratic Lagrangian. In this paper we examine the problem of N particles with spin that are coupled with strong bonds. The location of one particle is specified by the supervector

$$\hat{q}_k^i = q_k^i + \theta^\alpha q_{\alpha k}^i + \theta^1\theta^2 q_{12\,k}^i, \quad k = 1, \ldots, N$$

characterizing the location and spinor degrees of freedom of a particle. It is important that θ^{α} and $q^{i}_{\alpha k}$ are assumed to be odd elements of the Grassman algebra Λ, and q^{i}_{k} and q^{i}_{12k} are even elements. In addition, it is assumed that θ^{α} can be included in the system of generating Λ and that there is involution in Λ that influences θ^{α} according to the formula $\theta^{*}_{1} = -\theta_2, \theta^{*}_{2} = \theta_1$. It is also assumed that all $\hat{q}_k$ can be obtained from certain fixed initial positions $\hat{q}_{k0}$ by the same rotation in superspace

$$\hat{q}_k(t) = \hat{R}(t)\,\hat{q}_{0k}\,, \tag{1}$$

where the $\hat{R}(t)$ matrix has the form

$$\hat{R}_{ij} = R_{ij} + R^{\alpha}_{ij}\,\theta^{\alpha} + R^{12}_{ij}\,\theta^1\theta^2, \tag{2}$$

and the following conditions are satisfied: 1) R^{12}_{ij} are even elements of Λ, R^{α}_{ij} are odd elements of Λ, and R_{ij}, R^{α}_{ij}, and R^{12}_{ij} do not depend on θ^1 and θ^2; 2) $R^{+}R = 1$, where $R^{+}_{ij} = R^{*}_{ij}$. The R_{ij} matrices form a group which we will denote by SO_{Λ} (3). Such construction was examined earlier elsewhere.[3,5]

It is easy to see that this approach is analogous to the description of a classical solid—a spinning top. Within the context of spin systems this means that the relaxation time of the spins of individual particles is much shorter than the characteristic time of the entire system.

It is assumed that the action for an N-particle system with strong coupling has the form

$$S = \frac{m}{2}\sum_{k=1}^{N} \dot{\hat{q}}_k\,\dot{\hat{q}}^{*}_k\,. \tag{3}$$

To allow for the coupling (2) we introduce the gauge field

$$W = \hat{R}^{-1}(t)\,\dot{\hat{R}}(t)$$

which is a superanalog of the angular velocity. It is important that W belongs to the Lie algebra of the SO_{Λ} group (3). The following equality holds

$$W = \sum_{1}^{3} W_a f_a\,, \quad (f_a)_{bc} = -\epsilon_{abc}\,,$$

where W_a, $a = 1,2,3$ are even superfields. Using this equality, we can write Eq. (3) in the form

$$S = \sum_{ab=1}^{3} \hat{I}_{ab}\,W_a W_b\,,$$

where $\hat{I}_{ab}$ is the inertia supertensor, i.e.,

$$\hat{I}^{*}_{ab} = I_{ba}\,, \quad \hat{I}_{ab} = I_{ab} + \theta^{\alpha} I^{\alpha}_{ab} + \theta^1\theta^2 I^{12}_{ab}\,.$$

The equations of motion of the system assume the form of Euler equations for a spinning top

$$\dot{M}_a + \epsilon_{abc} W_b M_c = 0. \quad (4)$$

Here M_a is the supermoment and $M_a = I_{ab} W_b$.

It is important that the Euler equations (4) and the structure Hamiltonian corresponding to them (Refs. 2 and 4) are consistent with the Grassman structure. According to Ref. 2, to obtain the Euler equations (4), we must determine the Poisson brackets for the components of the supermoment in the following way:

$$\{M_a\ M_b\} = \epsilon_{abc} M_c. \quad (5)$$

The Poisson brackets (5) uniquely define the structure Hamiltonian of the model. The obtained equations, which agree with the ordinary Euler equations, have two integrals

$$M^2,\ E = I_{ab} W_a M_b.$$

The integration of Eqs. (4) is based on the following general consideration. Assume that there is a differential equation $\dot{x} = f(x)$, where $f(x)$ is a function of the real variable and x is an even element of the Grassman algebra $x = x_0 + \theta_\alpha x^\alpha + \theta_1\theta_2 X$. The solution reduces to solving the real equation in the domain of real x. To solve the equation in Grassman algebra, we must substitute in the obtained solution the even elements of Grassman algebra as the initial data. In the examined case the solution can be formed by transforming the elliptic integral via $\mathrm{sn}(\tau,k)$. After expanding $\mathrm{sn}(\tau,k)$ in terms of θ_1 and θ_2, we obtain the solution in a more detailed form. Here τ and k are defined by the formulas[4]

$$\tau = t\sqrt{\frac{(I_3 - I_2)(M^2 - 2EI_1)}{I_1 I_2 I_3}},$$

$$s = \Omega_2 \sqrt{\frac{I_2(I_3 - I_2)}{2EI_3 - M^2}}. \quad (7)$$

Using Eq. (7), we can calculate the remaining characteristics of the spinning supertop.

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