$B(0,1)$ superalgebra and explicit integration of the supersymmetrical Liouville equation

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(Submitted 31 May 1980
Pis'ma Zh. Eksp. Teor. Fiz. **32**, No. 1, 85–88 (5 July 1980)

A supersymmetrical Liouville equation with anticommuting spinor fields is integrated in explicit form and its general solutions, which are defined by four arbitrary functions, are formulated. The "potentials" of the gauge field in the zero-curvature representation in this case are functions that are important in the $B(0,1)$ [OSp(2,1)] superalgebra.

PACS numbers: 02.10.Nj, 01.30.Pj,

1. The methods of integrating nonlinear dynamic systems associated with graduated algebras, which were developed in previous papers,[1] allow generalization to the supersymmetrical case. In this case the odd elements of the corresponding superalgebras can be compared with the anticommuting (spinor) fields that have importance in Grassman superalgebra. In this paper we discuss in detail the supersymmetrical generalization of the Liouville equation associated with $B(0,1)$[OSp(2,1)]-type superalgebra and carefully examine the differences arising in the integration of supersymmetrical equations, in contrast to the usual case.

2. The supersymmetrical Liouville equation corresponding to the action $\int dz_+ dz_- d\theta_+ d\theta_- [-\frac{1}{2}\hat{\Phi}\hat{D}_-\hat{D}_+\hat{\Phi} + \exp\hat{\Phi}]$ has the form

$$\hat{D}_-\hat{D}_+\hat{\Phi} = \exp\hat{\Phi}, \tag{1}$$

where $\hat{\Phi} \equiv \hat{\Phi}(z_\pm, \theta_\pm) = \rho(z_\pm) - \bar{\theta}\omega(z_\pm) - \frac{1}{2}\bar{\theta}\theta F(z_\pm)$ is a superscalar field consisting of two scalar fields ρ and F and the Majorana spinor $\omega^\pm$ of the functions with anticommuting values, which depends on the $z_\pm$ coordinates of a two-dimensional space and Grassman variables $\theta \equiv (\theta_+, \theta_-); \bar{\theta} \equiv (-\theta_-, \theta_+)$. The superdifferentiation operators are $\hat{D}_\pm = \mp\partial/\partial\theta_\pm + \theta_\pm \partial/\partial z_\pm$; $\hat{D}^2_\pm = \mp\partial/\partial z_\pm$, $\hat{D}_+\hat{D}_- = -\hat{D}_-\hat{D}_+$ Equation (1) in the of the superfield $\hat{\Phi}$ $(F = \exp\rho)$ has the form

$$\rho_{,z_+z_-} = \exp 2\rho + \exp\rho\,\omega^+\omega^-; \quad \omega^\pm_{,z_\mp} = \exp\rho\,\omega^\mp, \tag{2}$$

and in the case $\omega^\pm = 0$ it becomes an ordinary Liouville equation identical to that obtained earlier.[5]

3. The $B(0,1)$ superalgebra (see, for example, Ref. 2) consists of five elements h, $X_\pm$, $Y_\pm$ that satisfy the commutation relations

$$[h, X_\pm]_- = \pm 2X_\pm, \qquad [h, Y_\pm]_- = \pm Y_\pm, \qquad [X_+, X_-]_- = [Y_+, Y_-]_+ = h,$$

$$[X_\pm, Y_\pm]_- = 0, \qquad [X_\pm, Y_\mp]_- = Y_\pm, \qquad [Y_\pm, Y_\pm]_+ = \mp 2X_\pm. \tag{3}$$

Let us introduce the following operators $A_{\pm}$ that are important in the $B(0,1)$ algebra,

$$A_{\pm} = u^{\pm} h + \phi^{\pm} X_{\pm} + \psi^{\pm} Y_{\pm}, \tag{4}$$

where $u^{\pm}(z_+, z_-)$ and $\phi^{\pm}(z_+, z_-)$ are the ordinary functions and $\psi^{\pm}(z_+, z_-)$ are the anticommuting functions, $(\psi^{\pm})^2 = \psi^+\psi^- + \psi^-\psi^+ = 0$. Thus, the "zero-curvature" representation for the $A_{\pm}$ [1] operators

$$[\partial/\partial z_+ + A_+,\ \partial/\partial z_- + A_-] = 0, \tag{5}$$

leads to the system

$$u^-_{,z_+} - u^+_{,z_-} + \phi^+\phi^- + \psi^+\psi^- = 0; \quad \phi^{\mp}_{,z_{\pm}} = \pm 2u^{\pm}\phi^{\mp};$$

$$\psi^{\mp}_{,z_{\pm}} \mp u^{\pm}\psi^{\mp} = \phi^{\mp}\psi^{\pm}, \tag{6}$$

which, after the obvious replacement of the variables $\phi^+\phi^- \equiv \exp 2\rho$ and $\psi^{\pm} \equiv \omega^{\pm}(\phi^{\pm})^{1/2}$, reduces to Eqs. (2).

4. The representation (5) is the condition for the gradient $A_{\pm}$ operators, i.e.,

$$A_{\pm} = g^{-1} g_{,z_{\pm}}, \tag{7}$$

where g is an element of the complex shell of the supergroup[3] G with generators (3), which can be represented in the form of a Gaussian expansion

$$g = M^+ N^- \exp H = M^- N^+ \exp H', \tag{8}$$

in which $M^{\pm}$ and $N^{\pm}$ are elements of the complex shells of the maximum nilpotent subgroups of G that are stretched along $X_{\pm}$ and $Y_{\pm}$, and H (H') belong to the Cartan subalgebra of G. Henceforth, for simplicity, we use the gauge $H' = 0$ in which $u^+ = 0, \phi^-{}_{,z_{\pm}} = 0$. It follows from Eqs. (4), (7), and (8) that the elements $M^{\pm}$ can be represented in the form $M^{\pm} = \exp(m^{\pm} X_{\pm} + \epsilon^{\pm} Y_{\pm})$, where $m^+(z_+)$, $m^-(z_-)$ and $\epsilon^+(z_+)$, $\epsilon^-(z_-)$ are, respectively, the ordinary and anticommuting functions of the arguments (compare with Ref. 1). The identity $(M^{\pm})^{-1}M^- = N^- \exp H (N^+)^{-1}$ from Eq. (8) makes it possible to determine the group parameters of the elements $N^{\pm} \equiv \exp(\tilde{m}^{\pm} X_{\pm} + \tilde{\epsilon}^{\pm} Y_{\pm})$ and $\exp H \equiv \exp(rh)$ in terms of the arbitrary functions $m^{\pm}$ and $\epsilon^{\pm}$ that parametrize $M^{\pm}$; specifically,

$$\exp(-r) = 1 - m^+ m^- - \epsilon^+\epsilon^-, \quad \tilde{\epsilon}^{\pm} = (\epsilon^{\pm} + m^{\pm}\epsilon^{\mp}) \exp r,$$

$$\tilde{m}^{\pm} = m^{\pm} \exp r. \tag{9}$$

Substituting in Eq. (7), Eq. (8) with the elements $M^{\pm}$, $N^{\pm}$, and $\exp H$ taken from Eq. (9) and comparing it with Eq. (4), we obtain the following final equation for general solutions of the supersymmetrical Liouville equation

$$\phi^{\pm} = (m^{\pm}_{,z_{\pm}} \pm \epsilon^{\pm}\epsilon^{\pm}_{,z_{\pm}}) \exp[(1 \pm 1)\tau], \quad u = -(\epsilon^{+}\epsilon^{-}_{,z_-} + m^{+}m^{-}_{,z_-}) \exp\tau,$$

$$\psi^{+} = \lambda^{-1}\epsilon^{+}_{,z_+} + \lambda^{-2} m^{+}_{,z_+}(\epsilon^{-} + \epsilon^{+} m^{-}), \quad \lambda \equiv 1 - m^{+}m^{-},$$

$$\psi^{-} = \epsilon^{-}_{,z_-}(1 - \lambda^{-1}\epsilon^{+}\epsilon^{-}) + \lambda^{-1}m^{-}_{,z_-}(\epsilon^{+} + \epsilon^{-} m^{+}),$$

$$\exp 2\rho = (m^{+}_{,z_+} + \epsilon^{+}\epsilon^{+}_{,z_+})(m^{-}_{,z_-} - \epsilon^{-}\epsilon^{-}_{,z_-}) \exp 2\tau . \tag{10}$$

5. The method used above to integrate the supersymmetrical Liouville equation (1) can be generalized in the standard way for arbitrary graduated superalgebras. In this case the basic problem is the formulation for the elements $N^{\pm}$ and $\exp H$ from Eq. (8) using the known $M^{\pm}$ that satisfy, as in the case of "ordinary" graduated algebras, the S-matrix-type equations. To solve the nonlinear equations associated with graduated-algebras characterized by the Cartan matrix (usually generalized),[4] we must know[1] the $\exp H$ element from Eq. (8), whose parameters can be determined by calculating the matrix elements of the known operator $(M^{+})^{-1}M^{-}$ between the highest and the lowest basis vectors. However, a simple example of the supersymmetrical Liouville equation shows that calculation of the highest vector of the element $(M^{+})^{-1}M^{-}$ equal to $1 - m^{+}m^{-} - \epsilon^{+}\epsilon^{-}$ is insufficient for describing the complete solution of Eq. (10) of the system (1).

We note that the supersymmetrical generalizations of the sine-Gordon equations (see, for example, Ref. 5), $\rho_{,z_+z_-} = 2\exp\rho - 2\exp(-\rho)$ and $\rho_{,z_+z_-} = 2\exp\rho - \exp(-2\rho)$, which have a nontrivial group of internal symmetries in ordinary space,[6] are apparently associated with finite-size superalgebras with the Dynkin schemes () and () and can be integrated just as in Ref. 7. All of the questions require further study.

In conclusion, the authors thank B. A. Arbuzov, A. A. Kirillov, P. P. Kulish, Yu. I. Manin, M. A. Mestvirishvili, and O. A. Khrustalev for useful discussions.

[1)]The matrix realization of the Lax representation for (1) in Ref. 5 corresponds to the special $B(0,1)$ representation within the context of our approach.

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