Helical instability of plasma with finite conductivity

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(Submitted 14 May 1980)
Pis'ma Zh. Eksp. Teor. Fiz. **32**, No. 2, 100–103 (20 July 1980)

It is shown that when the Kruskal-Shafranov condition $nq > m$ is satisfied in a plasma having finite conductivity there is a helical instability with an increment larger than that for the Thirring mode.

PACS numbers: 52.35.Py, 52.55.Gb, 52.55.Ez

According to the current theoretical understanding, the main large-scale instabilities in a Tokamak plasma are a helical mode described by ideal hydrodynamics and the Thirring mode, which is a manifestation of the helical mode in the presence of a resonant surface in the plasma. We shall show that there is yet another helical mode which does not reduce to either of these and which, in particular, has a larger increment than the Thirring mode.

We examine a cylindrical plasma pinch of radius a with a uniform current and a conductivity σ, located inside a sheath of radius b. Between the plasma and the sheath there is a region of zero conductivity. This model, which permits an exact investigation of stability, was proposed in Refs. 1 and 2, wherein the basic relationships and the total dispersion relation were given for perturbations of the type $\vec{\xi} = \vec{\xi}(r)e^{im\theta + ikz + \gamma t}$. Our problem, in contrast to that of Ref. 2, is to investigate this relation for the case of large conductivity. We assume that $ka = (a/R)n \ll 1$ (R is the large radius of an equivalent torus).

It is convenient to treat the dispersion relation of Ref. 2 as the equating to zero of a third order determinant whose columns are of the form

$$\left(\frac{2}{g\beta_i} - 1 - \beta_i\frac{\Gamma^2}{2g}\right)\left(\frac{ka}{m}\sqrt{1-\beta_i^2}\,\frac{\phi_i'}{\phi_i} + \beta_i\right),$$

$$1 - \beta_i^2 + \lambda\left(\frac{ka}{m}\sqrt{1-\beta_i^2}\,\frac{\phi_i'}{\phi_i} + \beta_i\right) \tag{1}$$

$$\frac{2}{g\beta_i} - \frac{2\beta_i}{g} + \lambda\left(\frac{ka}{m}\sqrt{1-\beta_i^2}\,\frac{\phi_i'}{\phi_i} + \beta_i\right), \quad i = 1, 2, 3$$

Here $g = m + ka\dfrac{B_s}{B_\theta} = m - nq(a)$, q is the margin of stability of the Tokamak, $\lambda = (b^{2m} + a^{2m})/(b^{2m} - a^{2m})$, Γ is the increment divided by the Alfvèn frequency along the field of the current: $\Gamma^2 = \gamma^2 \frac{4\pi\rho a^2}{B_\theta^2(a)} = \frac{\gamma^2}{\omega_0^2}$, and $\phi_i = I_m ka\sqrt{((1-\beta_i^2))}$ is a Bessel function of imaginary argument. The quantities β_1, β_2, and β_3, which are responsible for the columns being different, are solutions of the cubic equation:

$$\frac{1}{\Gamma\omega_0\tau}\beta^3 + \left(1 + \frac{g^2}{\Gamma^2}\right)\beta - \frac{2g}{\Gamma^2} = 0,$$
$$\tau = \frac{4\pi\sigma}{c^2k^2}. \tag{2}$$

We are interested in the case $\omega_0\tau \gg 1$ (for a Tokamak this quantity is of order 10^6). If $g \neq 0$, the solutions of Eq. (2) are, to sufficient accuracy for our purposes

$$\beta_1 = \frac{2g}{\Gamma^2 + g^2} \sim 1; \qquad \beta_2 = il - \frac{\beta_1}{2}, \qquad \beta_3 = -il - \frac{\beta_1}{2}, \tag{3}$$

$$l = \left|\sqrt{\omega_0\tau(\Gamma + g^2/\Gamma)}\right| \gg 1.$$

In view of the scale of β_1 and I, the Bessel functions can be simplified to $\frac{\phi_1'}{\phi_1} ka\sqrt{(1-\beta_1^2)} = 1$, $\phi_{2,3}'/\phi_{2,3} = 1$. The columns with $i=2$ and $i=3$ are complex conjugates. It is convenient to replace one of them by the imaginary and the other by the real part. In addition, we divide the first column by $(1+\beta_1)$. Then, retaining the leading terms in I, we obtain the dispersion relation

$$\begin{vmatrix} \frac{2}{g\beta_1} - 1 - \beta_1\frac{\Gamma^2}{2g} & -l - l^2\frac{\Gamma^2}{2g}\frac{ka}{m} & l^2\frac{\Gamma^2}{2g} \\ 1 + \lambda - \beta_1 & 2l\beta_1 + \lambda l & l^2 \\ \frac{2}{g\beta_1}(1-\beta_1) + \lambda & -\frac{2l}{g} + \lambda l & \lambda\frac{ka}{m}l \end{vmatrix}. \tag{4}$$

For hydromagnetic increments, when $\Gamma^2 \sim 1$, in the second column only the uppermost term contributes to the determinant; in the third column, only the two lower terms. The dispersion relation reduces to the equality to zero of the bottom term in the first column; this gives the following for the increment of the helical instability of an ideal plasma:

$$\Gamma^2 = g[2 - g(1+\lambda)]. \tag{5}$$

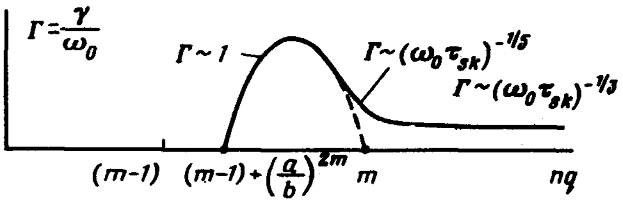
We now examine the case in which $I\frac{\Gamma^2}{2g} \sim 1$. Then the uppermost terms in the first $\left(\frac{2g}{\beta_1}-1=\frac{\Gamma^2}{g^2}\ll 1\right)$ and third columns are small compared to the other terms in these columns, and the dispersion relation reduces to the vanishing of the uppermost term in the second column:

$$l\frac{\Gamma^2}{2g}\frac{ka}{m}+1=0. \tag{6}$$

The increment Γ is real for $g=m-nq<0$, i.e., when the Kruskal-Shafranov condition is satisfied. The increment γ is equal to

$$\gamma=\omega_0\left(\frac{4m^2}{\omega_0\tau_{sk}}\right)^{1/3}, \qquad \omega_0^2=\frac{B_\theta^2(a)}{4\pi\rho a^2}, \qquad \tau_{sk}=\frac{4\pi\sigma a^2}{c^2}. \tag{7}$$

Taking into account the relation[2] $\gamma=\omega_0\left(\frac{16m^2}{\omega_0\tau_{sk}}\right)^{1/5}$ for $g=0$, one finds that the stability diagram of an individual helical harmonic in a plasma with finite conductivity looks like the following:



The dashed line shows the increment of the mode for infinite conductivity.

The mode obtained here is a manifestation of a helical instability in the region $nq>m$, where the ideal mode is stabilized by surface currents which arise under deformation of the plasma. These currents are not in equilibrium in the direction tangential to the boundary of the plasma, and this causes inertial flow in a boundary layer of thickness $\frac{a^2}{Rq}(2m\omega_0\tau_{sk})^{-1/3}$. The sheath has no effect on this mode, even if it coincides with the boundary of the plasma. A formal paradox is eliminated here, since the boundary is usually fixed by equating the normal velocity v_r of the plasma to zero. If there is a vacuum space the limit $b\to a$ means that the radial component of the magnetic-field perturbation $\widetilde{B}_r$ is zero, and this does not reduce to the condition $v_r=0$ for a plasma of finite conductivity.

The existence of our mode does not rely on the model chosen for the current distribution; the condition for its occurrence is simply that $nq>m$ at the boundary of the region in which the current is flowing. In our view, this explains why there is a peaked current distribution in Tokamaks, with practically all the current flowing in the central region, where $q<2$. It also explains why measures which promote peaking of the current in the initial stage permit the avoidance of MHD instabilities as the discharge builds.

The author is grateful to V. D. Shafranov for helpful advice while this work was in progress.

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[2]S. N. Breus, Zh. Tekh. Fiz. **30**, 1030 (1960) [Sov. Phys. Tech. Phys. **5**, 960 (1960)].