# Hidden symmetry of integrable systems

A. A. Belavin
*L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR*

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We formulate a principle of invariance with respect to a discrete subgroup of the Lorentz group of a 1 + 1 dimensional space which operates independently on states with different momenta.

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In this letter we examine the triangle equations (or the factoring equations, or the Yang-Baxter equations). These are the most compact encoded expressions of the hidden symmetry of one-dimensional quantum-mechanical or classical integrable systems[1–12] and two-dimensional lattice-statistical models of the Baxter type.[4–8,10] In this letter we formulate a principle of invariance with respect to a discrete subgroup of the Lorentz group of a 1 + 1 dimensional space which acts independently on particle states with different momenta. It is shown that this symmetry implies that the triangle equations are satisfied for the two-particle $S$ matrix describing the scattering of the particles.

The triangle equations were first obtained by Yang[3] in a study of the relativistic $n$-particle problem with a delta-function interaction, where they ensured the self-consistency of the Bethe ansatz. The same equations arise in factored relativistic scattering theory[9–11] and in the two-dimensional statistical lattice models of the Baxter type.[4–10] The triangle (Yang-Baxter) equations play a key role in the quantum-mechanical inverse-scattering-problem method developed by Faddeev *et al.*,[7,8] and they are also connected with the classical systems that are integrable by the inverse–scattering–problem method.[13] We shall write the triangle equations in the language of scattering theory for $N$ different types of particles:

$$S_{i_1 i_2}^{k_1 k_2}(u_1 - u_2)\, S_{k_1 i_3}^{j_1 k_3}(u_1 - u_3)\, S_{k_2 k_3}^{j_2 j_3}(u_2 - u_3)$$

$$= S_{i_2 i_3}^{k_2 k_3}(u_2 - u_3)\, S_{i_1 k_3}^{k_1 j_3}(u_1 - u_3)\, S_{k_1 k_2}^{j_1 j_2}(u_1 - u_2). \tag{1}$$

Here $S_{i_1 i_2}^{j_1 j_2}(u_1 - u_2)$ is a two-particle $S$ matrix; $i_\alpha(j_\alpha)$ denotes the type of the initial (final) particles and assumes values from one to $N$. The repeated indices $k_\alpha$ in Eq. (1) are summed over from one to $N$. The quantities $u_\alpha$ are the speeds of the three colliding particles and are related to the energy and momentum by the relations $E = mchu$ and $p = mshu$. Any nontrivial solution of equations (1) will lead to the existence of integrable systems. The question is how to discover these solutions. Meanwhile, it seems surprising at first glance that such solutions exist at all, because the number of equations in (1) is $N^6$, much larger than the number of unknown functions, which is $N^4$.

For equal speeds of the particles $u_1 = u_2 = u_3$, Eq. (1), has in addition to the trivial solutions $S^{12}(0) = 1$, a solution corresponding to total reflection of the particles: $S^{12}(0) = P^{12}$, where $P_{i_1 i_2}^{j_1 j_2} = \delta_{i_1}^{j_2} \delta_{i_2}^{j_1}$. The value of $P^{12}$ is unchanged during isotopic transformations of both particles before and after scattering

$$P_{i_1 i_2}^{j_1 j_2} = (G^{-1})_{i_1}^{i_1'} (G^{-1})_{i_2}^{i_2'} P_{i_1' i_2'}^{j_1' j_2'} G_{j_1'}^{j_1} G_{j_2'}^{j_2} \tag{2}$$

where $G$ is any $N \times N$ matrix. We shall examine a whole-number lattice in the complex plane of speeds $u_{\mathbf{k}} = k_1 + \tau k_2$, where $\mathbf{k}(k_1 k_2)$ is a whole-number lattice vector and $\tau$ is a complex number ($\mathrm{Im}\tau > 0$). We place an $N \times N$ dimensional matrix $G_{\mathbf{k}}$ in correspondence with each point $\mathbf{k}$ of the lattice. We require that $G_{\mathbf{k}}$ form a representation (projection) of the Abelian group of displacements on the lattice, i.e., that it satisfy up to a multiplicative factor the relations

$$G_{\mathbf{k}} G_{\mathbf{e}} = G_{\mathbf{k}+\mathbf{e}}; \quad G_{\mathbf{k}}^{-1} = G_{-\mathbf{k}}; \quad G_{o} = 1. \tag{3}$$

Using the matrix $G_{\mathbf{k}}$, one can construct a solution of Eq. (1) at all points on the lattice:

$$S_{i_1 i_2}^{j_1 j_2}(u_{\mathbf{k}}) = \left(G_{\mathbf{k}}^{-1}\right)_{i_1}^{i_1'} P_{i_1' i_2}^{j_1' j_2} \left(G_{\mathbf{k}}\right)_{j_1'}^{j_1}. \tag{4}$$

By virtue of the group properties (3),$G_{\mathbf{k}}$ is of the form $G_{\mathbf{k}} = g^{k_1} h^{k_2}$. Here $g$ and $h$ are matrices corresponding to the two elementary displacements. Equations (3) imply the commutation relations $gh = \omega hg$ ($\omega$ is a numerical factor), which ensure the path-independence of $G_{\mathbf{k}}$ between the origin and point $\mathbf{k}$. It remains to recover an analytic function $S^{j_1 j_2}_{i_1 i_2}(u)$ over the entire plane. It follows from Eq. (4) that

$$S^{j_1 j_2}_{i_1 i_2}(u_{\mathbf{k}} + 1) = (g^{-1})^{i_1'}_{i_1} S^{j_1' j_2}_{i_1' i_2}(u_{\mathbf{k}})\, g^{j_1}_{j_1'} = g^{i_2'}_{i_2} S^{j_1 j_2'}_{i_1 i_2'}(g^{-1})^{j_2}_{j_2'}$$

$$S^{j_1 j_2}_{i_1 i_2}(u_{\mathbf{k}} + \tau) = \lambda (h^{-1})^{i_1'}_{i_1} S^{j_1' j_2}_{i_1' i_2}(u_{\mathbf{k}})\, h^{j_1}_{j_1'} = \lambda h^{i_2'}_{i_2} S^{j_1 j_2'}_{i_1 i_2'}(h^{-1})^{j_2}_{j_2'}. \tag{5}$$

We require that the matrix $S^{12}(u)$ have the same properties for any $u$. That is, we drop the indices:

$$S^{12}(u + 1) = g_1^{-1} S^{12}(u)\, g_1 = g_2 S^{12}(u)\, g_2^{-1}, \tag{6}$$

$$S^{12}(u + \tau) = \lambda \exp(2\pi i u)\, h_1^{-1} S^{12}(u)\, h_1 = \lambda \exp(2\pi i u)\, h_2 S^{12}(u)\, h_2^{-1}. \tag{7}$$

The two-particle $S$ matrix is uniquely determined by the initial condition and by the requirement that it be automorphic (Eq. (7)). The solution of Eq. (7) can be found by expanding in a Fourier series. In order that the function $S^{12}(u)$ constructed in this way be a solution of Eq. (1) for all values of $u$, the matrices $g$ and $h$ must satisfy some conditions which are as yet in general unknown; we shall demonstrate this by an example.

Recall that the argument of the $S$ matrix is equal to the difference $u = u_1 - u_2$ between the speeds of the colliding particles. Therefore, Eq. (7) represents the requirement that the two-particle $S$ matrix be invariant under a discrete Lorentz transformation of one of the initial and one of the final particles with one of the speeds (for example, $u_1$), accompanied by an exchange of these particles for particles of the other kind (transformation with $g$ and $h$). We stress that the momentum and kind of the second initial and second final particle do not change during this transformation.

We now examine an example for which one can explicitly carry out the construction described above. We take $g$ and $h$ to be matrices for which $g^N = h^N = 1, \omega = \exp(2\pi i/N)$. They can be chosen in the form

$$g = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \ddots \; \omega^{N-1} \end{pmatrix}; \qquad h = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & 1 \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix}. \tag{8}$$

It will be convenient to introduce a complete set of $N \times N$ matrices: $I_{\vec{a}} \equiv I_{a_1 a_2} = g^{a_1} h^{a_2}; a_{1,2} = 0, \ldots, N-1$. Then, with allowance for the second equations in (6) and (7) ($Z_N \times Z_N$ invariance), the $S$ matrix assumes the form

$$S_{i_1 i_2}^{j_1 j_2}(u) = \sum_{\vec{a}} W_{\vec{a}}(u)\,(I_{\vec{a}})_{i_1}^{j_1}\,(\bar{I}_{\vec{a}})_{i_2}^{j_2}. \tag{9}$$

Here the superior bar denotes Hermitian conjugation. The automorphicity conditions (6) and (7) and the initial condition, after substitution of Eq. (9), assume the form

$$W_{\vec{a}}(u+1) = \omega^{a_2} W_{\vec{a}}(u)\,;\quad W_{\vec{a}}(u+\tau) = \lambda \exp(2\pi i u)\,\omega^{a_1} W_a(u), \tag{10}$$

$$W_{\vec{a}}(0) = 1. \tag{11}$$

The solution is obtained by expanding in a Fourier series. It is of the form

$$W_{\vec{a}} = \Theta_{\vec{a}}(u+\eta)\,/\,\Theta_{\vec{a}}(\eta). \tag{12}$$

Here

$$\Theta_{\vec{a}}(u) = \sum_{m=-\infty}^{\infty} \exp\left[i\pi\left(m+\frac{\vec{a}_2}{N}\right)^2 \tau + 2\pi i\left(m+\frac{a_2}{N}\right)\left(u+\frac{a_1}{N}\right)\right].$$

For $N = 2$ this solution coincides with that examined by Baxter in Refs. 5 and 6. In the same way, one can construct other solutions as well, including solutions in the form of $\Theta$ functions of several variables. From the example examined above, it can be seen that Eqs. (6) and (7) reduce the number of independent equations in (1) to the number of independent functions. A proof that Eq. (12) is a solution of (1) for any $u$ will be published elsewhere.

A more interesting question concerns multidimensional (Yang-Mills?!)[1)] integrable systems. In a pioneering work Zamolodchikov[12] examined three-dimensional systems with properties analogous to Eq. (1). In these systems the role of the particles is played by infinite relativistic strings, undergoing trinary collisions in a 2 + 1 dimensional space. The role of the triangle equations is played by tetrahedral equations for the three-string collision amplitudes $S^{123}(n_1,n_2,n_3)$. The amplitudes depend in a Lorentz-invariant way on the unit vectors $n_i$ normal to the world planes of the propagation of the strings. Although the number of tetrahedral equations (8 thousand) is even still higher than the number of amplitudes $S^{123}$, they nevertheless have a simultaneous solution.[12] It is possible that this "miracle" is due to the invariance of the tetrahedral equations and the automorphicity of $S^{123}(n_1 n_2 n_3)$ with respect to the operation of some discrete Lorentz subgroup of 2 + 1 space.

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