# N-soliton solutions of Einstein-Maxwell equations

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The *L-A* pair, corresponding to the Einstein-Maxwell equations, is found for the case in which the space-time metrics and the 4-potential of the electromagnetic field depend only on two coordinates, and the *N*-soliton solutions are formulated.

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The methods of the inverse scattering problem, used in Refs. 1,2 for constructing exact (soliton) solutions of Einstein's equation in vacuum with metrics, depending only on two coordinates, and permitting generalization to the case in which an electromagnetic field is present with the invariant[3] $F_{ik}F^{ik}=0$, are also applicable in the general case of the presence in space of gravitational and electromagnetic fields whose metrics and 4-potential have the form.

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu + g_{ab}\,dx^a dx^b, \quad A_i = \{0,\,0,\,A_a\}, \tag{1}$$

where $\mu,\nu,... = 0,1; a,b,... = 2,3$; and the functions $g_{\mu\nu}, g_{ab}$ and $A_a$ depend only on $x^\mu$ and satisfy the Einstein-Maxwell equations without any additional restrictions on the form of these fields.

By a coordinate transformation of $x^\mu$ (without the participation of $x^a$)$g_{\mu\nu}$ can be reduced to the conformally two-dimensional form $g_{\mu\nu} = -f\eta_{\mu\nu}$, where $f>0$, and $\eta_{\mu\nu}$ = const. For stationary axisymmetric fields in cylindrical coordinates $x^\mu = \{\rho,z\}$, $x^a$ $\{t,\phi\}$, we must set $\eta_{\mu\nu} = \mathrm{diag}(1,1)$. For fields depending on the time and one spatial coordinate, i.e., for $x^\mu\{t,x\}$ and $x^a = \{y,z\}$ we can choose, for example $\eta_{\mu\nu} = \mathrm{diag}(-1,1)$.

The Einstein-Maxwell equations (for $\gamma = c = 1$)

$$R_i^k = -2\left(F_{im}F^{km} - \frac{1}{4}\delta_i^k F_{em}F^{em}\right),\ \nabla_m F^{km} = 0,\ F_{ik} = \partial_i A_k - \partial_k A_i \tag{2}$$

($R=0$ for electrovacuum fields) separate into two groups of equations when (1) is taken into account, giving a closed system for $g_{ab}$ and $A_a$ and equations that define $f$ in quadratures in terms of the found $g_{ab}$ and $A_a$.

The closed system of equations for $g_{ab}$ and $A_a$, obtained by the substitution of (1) into (2), can be written in an equivalent complex $3\times3$ matrix form[1)]

$$\eta^{\mu\nu}\partial_\mu U_\nu + \frac{i}{2a}\epsilon^{\mu\nu}U_\mu U_\nu = 0, \quad \epsilon^{\mu\nu}\partial_\mu U_\nu = 0, \tag{3}$$

$$\partial_\mu(G - 4i\beta\Omega) = -2(\Omega U_\mu - U_\mu^+\Omega), \tag{4}$$

  

$$GU_\mu = -4i\epsilon\, a\, \epsilon_\mu^{\ \nu}\, \Omega\, U_\nu, \tag{5}$$

where " + " denotes the Hermitian conjugate, $\epsilon = \pm 1$ is a sign, opposite to the sign of the determinant $\eta_{\mu\nu}$; $\epsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\epsilon_\mu^\nu = \eta_{\mu\sigma}\epsilon^{\sigma\nu}$ (the indices $\mu,\nu,...$ are lowered and raised by means of $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$; $\alpha$ is any (anisotropic) solution of the equation $\eta^{\mu\nu}\partial_\mu\partial_\nu\alpha = 0$, and $\beta$ is defined in terms of $\alpha$: $\partial_\mu\beta = \epsilon\epsilon_\mu^{\ \nu}\partial_\nu\alpha$. The matrices $G$ and $\Omega$ have the form

$$G = \left(\begin{array}{c|c} -4h^{ab} + 4\Phi^a\overline{\Phi}^b & -2\Phi^a \\ \hline -2\overline{\Phi}^b & 1 \end{array}\right), \quad \Omega = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$g_{ab} = -\epsilon_{ac}h^{cd}\epsilon_{db}, \quad A_a = -\epsilon_{ac}\,\mathrm{Re}\,\Phi^c \quad \text{и} \quad \epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We note that Eqs. (3) contain only the matrices $U_\mu (\mu = 0,1)$ as unknowns, Eqs. (4) actually serve to define the matrix $G$ in terms of $U_\mu$, and Eqs. (5) will then play the role of additional conditions on the choice of the solutions of (3).

In connection with the system (3) we consider the linear system (analog of the $L$-$A$ equations) for the complex $3\times 3$ matrix function $\Psi$, containing the additional complex parameter $\lambda$:

$$D_\mu \Psi = \Lambda_\mu^{\ \nu} U_\nu \Psi, \tag{6}$$

where, by analogy with Ref. 1, the operators $D_\mu = \partial_\mu + P_\mu(\partial/\partial\lambda)$; the functions $\Lambda_\mu^{\ \nu}, P_\mu$ and the matrix $\Psi$ depend on $x^\mu$ and $\lambda$, and the matrices $U_\mu$ only on $x_\mu$. If $P_\mu$ and $\Lambda_\mu^{\ \nu}$ are chosen in the form

$$P_\mu = -\left(\alpha\frac{\partial F}{\partial\lambda}\right)^{-1}\partial_\mu(\alpha F + \beta), \quad \Lambda_\mu^{\ \nu} = \frac{i}{2\alpha}\,\frac{\epsilon F\delta_\mu^\nu + \epsilon_\mu^{\ \nu}}{(1-\epsilon F^2)},$$

where $F$ is an arbitrary function of $x^\mu$ and $\lambda$, then the operators $D_\mu$ commute and the integrability conditions of the system (6) are identical with (3). It is convenient to choose $F(x^\mu,\lambda)$ such that $P_\mu = 0$. For this we set $\alpha F + \beta = \lambda$.

It is remarkable that with the introduction of this same complex parameter $\lambda$ in a natural manner the Eqs. (4) and anti-Hermitian parts of (5) can be represented in the form

$$D_\mu W + \Lambda_\mu^{\ \nu}(WU_\nu - U_\nu^+ W) = 0, \qquad W \equiv G + 4i(w-\beta)\Omega,$$

which is equivalent to the condition $\Psi^+ W \Psi = K(w)$, in which $K(w)$ is an arbitrary Hermitian matrix, depending only on $w \equiv \alpha F + \beta$.

Following Ref. 1 again, we introduce in place of $\Psi$ the new matrix variable $\chi: \Psi = \chi \mathring{\Psi}$, where $\mathring{\Psi}$ corresponds to some chosen exact solution $\mathring{U}_\mu$, $\mathring{W}$. Then we obtain for $\chi$.

$$D_\mu \chi = \Lambda_\mu{}^\nu ( U_\nu \chi - \chi \mathring{U}_\nu ), \quad \chi^+ W \chi = \mathring{W}. \tag{7}$$

In addition, it must be required that $\chi$ be regular in the vicinity of $w = \infty$ and $\chi(\infty) = I$.

The $N$-soliton solutions correspond to the meromorphic structure of $\chi$ and $\chi^{-1}$ in the $w$ plane:

$$\chi = I + \sum_{l+1}^{N} \frac{R_l}{w - w_l}, \quad \chi^{-1} = I + \sum_{l=1}^{N} \frac{S_l}{w - \widetilde{w}_l}. \tag{8}$$

Substitution of (8) into (7) gives the following results: the poles $\widetilde{w}_l$ ($l = 1,2,...,N$) complex conjugate to $w_l$; the matrices $R_l$ are degenerate:$R_l = n_l \times m_l$; the vectors $m_l$ are represented in the form $m_l = k_l M_l$, where $M_l = \mathring{\Psi}^{-1}(w_l)$, and $k_l$ (for each $l = 1,2...,N$) are arbitrary constant three-dimensional complex vectors; the vectors $n_l$ are determined from the algebraic system

$$\sum_{l=1}^{N} \Gamma_{kl} n_l = V_k \bar{m}_k, \quad \Gamma_{kl} = \frac{m_l V_k \bar{m}_k}{w_l - w_k}, \quad V_k = \mathring{W}^{-1} (\bar{w}_k).$$

Finally, for the $N$-soliton solutions we have

$$U_\mu = \mathring{U}_\mu + 2 i \partial_\mu R, \quad G = \mathring{G} - 4 i ( R^+ \Omega + \Omega R ), \quad R = \sum_{k=1}^{N} R_k.$$

Substitution of the solutions obtained in general form into the additional condition (5) transforms them into identities. Consequently, the solutions found satisfy the Einstein-Maxwell equations.

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[1)] The complex self-dual form of writing the equations, found in Ref. 4, was used in the derivation of Eqs. (3)-(5).

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