Internal structure of relativistic stars

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An exact general solution for equilibrium, spherically symmetric configurations is obtained within the context of the general theory of relativity. The dependence of the gravitational and thermal energies on the radius of a star is plotted. The luminosity of stars of the main sequence is discussed.

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Spherically symmetric equilibrium configurations are examined within the context of Einstein's general theory of relativity.[1]

1. EQUATIONS OF MOTION AND ENERGY RELATIONS

In the case of spherical symmetry when the space-time metric is determined by the expression $ds^2 = c^2dt^2 - a^2dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$, the equations of motion are represented in the form[2]

$$\frac{1}{a^2}\left(\frac{2a'}{ra} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \kappa\left(\frac{\rho W}{\Gamma^2} - p\right), \quad \frac{1}{a^2}\left(\frac{2c'}{rc} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \kappa\left(\frac{\rho W V^2}{\Gamma^2} + p\right),$$

$$\frac{1}{ac}\frac{\partial}{\partial t}\frac{\dot a}{c} - \frac{1}{a^2}\left(\frac{c''}{c} - \frac{c'a'}{ca} + \frac{c'}{rc} - \frac{a'}{ra}\right) = -\kappa p, \quad \frac{2\dot a}{ra^2} = -\kappa\frac{\rho W c V}{\Gamma^2}, \tag{1}$$

where the prime and the dot denote partial derivatives with respect to r and t, $W = 1 + \epsilon + p/\rho$, $\Gamma = \sqrt{1 - V^2}$, $V = a\dot r/c$, ρ and p are density and pressure in the rest frame, ϵ is the internal energy per unit rest mass, and the velocity of light $c \to 1$ as $r \to \infty$. Equations (1) comprise a system of four equations for five unknown functions c, a, ρ, p, and V. Introducing the entropy S, according to the thermodynamic identity $TdS = dW - dp/\rho$, we obtain the following expression as a corollary of (1):

$$\frac{1}{r^2}(r^2 G')' = \frac{\kappa}{2}\rho^*, \quad \frac{\partial \rho^*}{\partial t} + \mathrm{div}(\rho^* + p)\,\mathbf{v} = 0, \quad \frac{\partial}{\partial t}\frac{a\rho}{\Gamma} + \mathrm{div}\frac{a\rho}{\Gamma}\mathbf{v} + \frac{a\rho T}{\Gamma W}\frac{dS}{dt} = 0, \tag{2}$$

where

$$G' \equiv \frac{1}{2r}\left(1 - \frac{1}{a^2}\right), \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \dot r\frac{\partial}{\partial r}, \quad \mathrm{div}\, a\mathbf{v} \equiv \frac{1}{r^2}\frac{\partial}{\partial r} r^2 a\dot r, \quad \rho^* \equiv \frac{\rho W}{\Gamma^2} - p.$$

The first equation in (2) determines the gravitational potential G and the total energy density ρ^*, the second equation describes the law of conservation of the total energy, and the third equation determines the masses and the entropies. The last equation also holds for arbitrary metric $ds^2 = g_{ik}dx^i dx^k$ when it is written in the form

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\sqrt{-g}\,\rho u^i + \frac{\rho T}{W}u^i\frac{\partial S}{\partial x^i} = 0, \text{ where } u^i = \frac{dx^i}{ds}, \quad g = |g_{ik}| .$$

If the energy output q due to burning of the rest mass is known, then this equation can be broken down into two equations $\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\sqrt{-g}\rho u^i = q$ and $\frac{\rho T}{W}u^i\frac{dS}{dx^i} = -q$, and we obtain a total system of equations.

The total energy of a star M deduced from its gravitational field by an outside observer and the energy locked in the rest mass M_0, respectively, are

$$M = 4\pi\int_0^R \rho^* r^2\, dr, \quad M_0 = 4\pi\int_0^R \frac{a\rho}{\Gamma}r^2\, dr. \tag{3}$$

The energy M can be represented by the sum $M = M_0 + M_G + M_T + M_K$, where the gravitational energy M_G, thermal energy M_T, and kinetic energy M_K are determined by the expressions

$$M_G = 4\pi\int^R \rho^*(1-a\ )r^2 dr, \quad M_T = 4\pi\int_0^R a\left(\frac{\rho\epsilon + p}{\Gamma^2} - p\right)r^2 dr,$$

$$M_K = 4\pi\int_0^R \frac{a\rho}{\Gamma}\left(\frac{1}{\Gamma} - 1\right)r^2\, dr. \tag{4}$$

The energy emission of a source is evidently accompanied by a reduction of its mass M. M can vary by releasing (or absorbing) energy in nuclear and chemical reactions, which is described by the variation of M_0, and also because of variation of the partial energies M_G, M_T, and M_K.

The external gravitational field is described by Schwarzschild's solution[2]

$$c^2 = 1 - r_g/r, \quad a^2 = (1 - r_g/r)^{-1}, \quad r_g \equiv \kappa M/4\pi. \tag{5}$$

2. EQUILIBRIUM CONFIGURATIONS

As $V \to 0$ Eqs. (1) reduce to a system of three equations for four unknown functions c, a, p, and $\rho^* = \rho\epsilon + p$. The gravitational and thermal energies of the equilibrium configurations are expressed by the integrals

$$\frac{M_G}{M} = \frac{\kappa}{r_g}\int_0^R\left(1 - \frac{1}{\sqrt{1-\mu}}\right)\rho^* r^2 dr, \quad \frac{M_T}{M} = \frac{\kappa}{r_g}\int_0^R \frac{\rho\epsilon r^2\, dr}{\sqrt{1-\mu}}, \tag{6}$$

where $\mu = 1 - 1/a^2 = 2rG' = \kappa m/4\pi r$ and $m = 4\pi\int_0^r \rho^* r^2 dr$ is the moving mass.

(a) If we set $\rho^* = \text{const}$ as an additional equation, we shall obtain Schwarzschild's intrinsic solution

$$\frac{1 + 3p/\rho^*}{1 + p/\rho^*} = \sqrt{\frac{1 - r_g r^2}{1 - r_g}}, \quad a^2 = \frac{1}{1 - r_g r^2}, \quad c^2 = \frac{1 - r_g}{(1 + p/\rho^*)^2}, \tag{7}$$

which satisfies the boundary condition $p = 0$ when $r = R = 1$. This solution, however, has a discontinuity ρ at $r = R$. The Tolman model solutions also have analogous shortcomings as well as infinities at $r = 0$.[3]

Explicit expressions for M_G and M_T can be obtained for the solution of Eq. (7)

$$\frac{M_G}{M} = 1 - \frac{3}{2}\frac{\phi - ks}{s^{3/2}}, \quad \frac{M_T}{M} = \frac{3}{\gamma - 1}\frac{k}{s^2}\left(\frac{5}{2} + \frac{12k^2 - 1}{2ks}\phi - \frac{4}{s}\sqrt{9k^2 - 1}\right.$$
$$\left.\times \arctan\sqrt{\frac{3k + 1}{3k - 1}\frac{1 - k}{1 + k}}\right). \tag{8}$$

Here $s \equiv \sin\phi = \sqrt{r_g}$, $k \equiv \cos\phi > 1/3$, and $\rho\epsilon = p/(\gamma - 1)$.

(b) For a specified dependence $p = p(\rho)$, the problem reduces to a solution of the system of equations

$$p' = -r\frac{p + \rho^*}{2}\frac{\kappa p + \mu/r^2}{1 - \mu}, \quad (\mu r)' = \kappa\rho^* r^2, \tag{9}$$

for a relativistic degeneracy $\rho^* = \rho + 3p$, $\rho = \rho_0(T/T_0)^3$, $p = p_0(T/T_0)^4$, and $y = 4/3$.

(c) The general solution, which depends on one arbitrary function, can be obtained by assuming that the $c(x)$ function, where $x = r^2$, is specified. The equilibrium problem in this case reduces to the linear equation for $\mu = 1 - 1/a^2$,

$$\mu' + \frac{4xc'' - c/x}{c + 2xc'}\mu = \frac{4xc''}{c + 2xc'}, \tag{10}$$

whose solution is

$$\mu = e^{\gamma}\left[\mu(R^2) + 4\int_{R^2}^{r^2}\frac{e^{\gamma}c''xdx}{c + 2xc'}\right] \quad \gamma = \int_{R^2}^{x}\frac{4xc'' - c}{c + 2xc'}\frac{dx}{x}, \tag{11}$$

where $\mu(R^2) = r_g/R$. The requirement that c and c' must be continuous in matching them with the external solution (5) leads to the condition $p(R^2) = 0$ and the continuity of c'' leads to the condition $\rho^*(R^2) = 0$. If c are known and $a = 1/\sqrt{1 - \mu}$, then the values of p and ρ^* can be determined by using the formulas

$$\kappa p = \frac{4c'}{c}(1-\mu) - \frac{\mu}{x}, \quad \kappa\rho^* = \frac{8xcc''}{(xc^2)'}(1-\mu) + \left[1 + \frac{2c^2}{(xc^2)'}\right]\frac{\mu}{x}. \tag{12}$$

If $c^2(x)$ is defined as a relation of linear functions and the boundary conditions for the continuity of c, c', and c'' are satisfied for $r = R = 1$, then we obtain a particular solution $c^2 = 1 - r_g \dfrac{3' + x}{1' + 3x}$, which depends only on the parameter r_g

$$\mu = \frac{3h^2x}{1-n^2z^2}\left[z^2 + \frac{1-n^2z}{1-n^2z^2}\left(\frac{1-nz}{1+nz}\right)^n\left(\Delta + F\right)\right], \quad F = 2\int_{1/4}^{z}\left(\frac{1+nz}{1-nz}\right)^n z\,dz, \tag{13}$$

where $z = (1+3x)^{-1}$, $\Delta = (2-r_g)/32\,[(4+n)/(4-n)]^n$, $n^2 = 8r_g/(3-r_g)$. This expression is meaningless for $r_g > 1/3$ as $c(0)\to 0$. The $F(z)$ integral can easily be approximately calculated for $r_g \ll 1$ and $(1/3 - r_g) \ll 1$, and it is expressed by the elementary functions for $r_g = 1/11$.

Figure 1 shows the dependence of M_G, M_T, and $M_\Sigma = M_G + M_T$ on R/r_g for the equilibrium configuration (13) at $\rho\epsilon = p/(\gamma - 1)$, $\gamma = 4/3$, and $\gamma = 5/3$. The boundary value of R/r_g is determined by the condition $c(0)\to 0$; the infinite energy barrier prevents further decrease of R/r_g. Figure 2 shows the $M_\Sigma(R/r_g)$ plot constructed by means of a numerical integration of Eqs. (9) for $\gamma = 4/3$. The ambiguity is attributable to the fact that the radiation pressure, which gives rise to an additional parameter $\alpha = p_0/\rho_0$, is taken into account. The M_Σ function was obtained by Fowler for small

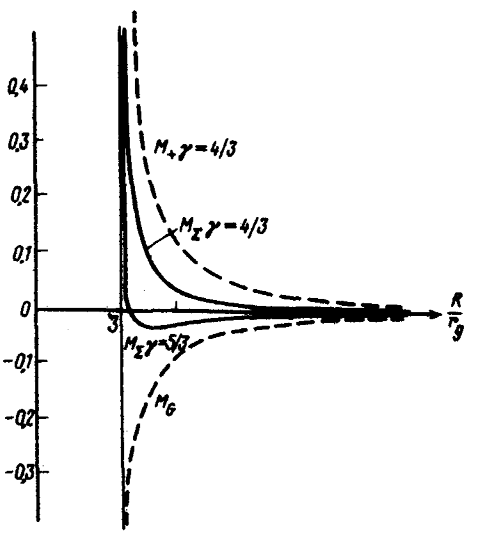

FIG. 1.

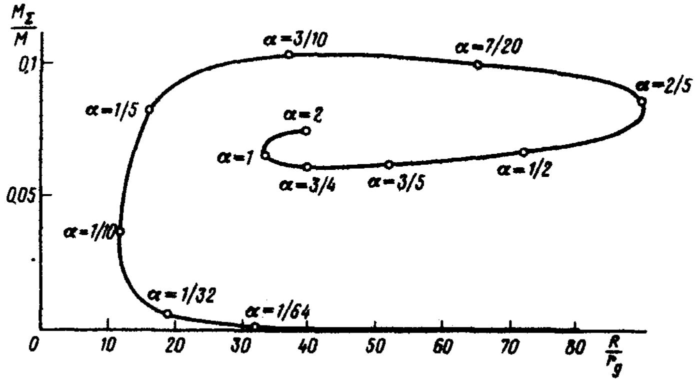

FIG. 2.

r_g/R.[4] The characteristic feature of the behavior of the energy M_Σ is its increase with decreasing R, which leads to a large release of the rest energy M_0.

3. LUMINOSITY OF STARS

To determine the luminosity $L = \pi R^2 \sigma T_1^4$, we must know the effective temperature T_1, which is apparently determined by the optical thickness of the star's atmosphere that coincides in order of magnitude with the barometric length $l = kT_1/m_A g$, where $g = GM/R^2$ and G is the Newtonian constant. Hence the opacity of the atmosphere is $\kappa_A = GM/p_1 R^2$. We shall limit ourselves to nonrelativistic Emden[5] equilibria $p = p_0(\rho/\rho_0)^{y_0}, y_0 = \text{const}$, which have a constant supply of convective stability if $y_0 < y$, where $\rho_0 \sim M/R^3$ and $T_0 \sim M/R$.

If $\kappa_A \sim \rho^\nu/T^\lambda$, $\nu = \text{const}$, and $\lambda = \text{const}$, then from the equilibrium conditions $T_1 \sim M^{a/c}/R^{b/c}$, where $a = (\nu + 1)/(\gamma_0 - 1) - \nu$, $b = (\nu + 1)/(\gamma_0 - 1) - 3\nu - 1$, and $c = (\nu + 1)/(\gamma_0 - 1) - \lambda + 1$. We shall require that the luminosity L must depend only on the mass M and must be independent of the radius R, as is the case for the main sequence stars $L \sim M^N$. Thus the parameters y_0 and N can be expressed in terms of ν and λ: $\gamma_0 = 1 + (1 + \nu)/(3 + 6\nu - \lambda)$, $N = 2(3 + 5\nu - \lambda)/(2 + 3\nu - \lambda)$.

If we use the approximation formula $\kappa_A \sim (\rho/T^{3.5})^\alpha$, then we shall obtain $\gamma_0 = 1.33$, 1.35, and 1.36 and $N = 3$, 4.3, and 6, respectively, for $\alpha = 0$, 0.5, and 1. It follows from this that the luminosity of the main-sequence stars is well described by the Emden equilibrium configuration with $\gamma_0 = 1.34$, which does not have any parameters, except M and R. We note that the Emden equilibrium with $\gamma_0 = 4/3$, which was proposed by Eddington on the basis of other premises, was considered a long time the standard for stars such as the sun.

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