Poisson brackets and continuous dynamics of the vortex lattice in rotating He II

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We determined the Poisson brackets for the macroscopic variables, which describe the states of He II in a rotating vessel. The hydrodynamics equations, which include the equations of the elasticity theory for the lattice of vortices, were obtained from the energy functional with the help of these brackets. The vibrational modes, including the Tkachenko waves, were also determined.

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A sufficiently fast rotation of He II produces in it a lattice of quantized vortices (the circulation quantum $\kappa = 2\pi\hbar/m$ where m is the mass of the He^4 atom), which on the average imitates the rigid-body rotation of the superfluid component of He II. The phenomenological theory of the vortex motion in a rotating He II, in which each volume element contains many vortices and the superfluid velocity v^s is equal to the average value of the velocity fields of the individual vortices, was formulated by Bekarevich and Khalatnikov.[1] This theory does not contain the additional mode associated with the vibrations of the lattice of the vortices,[2,3] which was determined later from microscopic calculations and which arises from the violation of translational symmetry in the presence of discrete vortices. In the limit $\kappa \to 0$ when the lattice of the vortices is converted to a system with a continuously distributed vorticity, this mode changes to regular inertial vibrations of the classical rotating liquid[4]; therefore, it is obtained from the theory only in this limiting case.[1] Our aim is to include the degrees of freedom associated with the lattice vibrations into the general theory of rotating He II. We limit ourselves to the case $T = 0$. A generalization to nonzero temperatures with allowance for dissipation will be published later. We use the method of Poisson brackets developed in Ref. 5. To obtain by this method the macroscopic dynamics of any condensed medium, we must know the energy functional of the system, which is expressed in terms of the hydrodynamic variables describing the state of the system and the Poisson brackets for these variables which are universal in nature.

In the rotating He II these variables are the density of the mass ρ, the velocity v^s, and the variables describing displacement of vortices in the lattice. In the continual description the nonequilibrium state of the vortices in a two-dimensional lattice can be determined by two functions X_1 and X_2, which are constant on the vortex line, rather than by three functions which determine the position of a lattice point in a three-dimensional crystal. It follows from the definition of these functions that $\vec{\nabla} X_\mu$ ($\mu = 1,2$) is perpendicular to the vortex line, i.e., there is a bond between the components of the velocity and X_μ:

$$\mathrm{rot}\, \mathbf{v}^s \vec{\nabla} X_\mu = 0\,. \tag{1}$$

If the displacements relative to the equilibrium state are small, then

$$X_1(\mathbf{r}) = x - u_x, \qquad X_2(\mathbf{r}) = y - u_y , \tag{2}$$

where $r = (x,y,z)$ is the Cartesian coordinate system with the z-axis directed along the rotational axis and $u = \hat{u}_x + \hat{y}u_y$ is a small transverse displacement of the vortices. The Hamiltonian of the rotating He II is expressed in terms of these variables as follows:

$$H = \int d^3r\, \epsilon(\rho, \mathbf{v}^s - [\,\boldsymbol{\Omega}, \mathbf{r}\,], \; g_{ik} - g^{\circ}_{ik}), \tag{3}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the vessel, g_{ik} is the metric tensor:

$$g_{ik} = \sum_{\mu=1,2} \frac{\partial X_\mu}{\partial x_i} \frac{\partial X_\mu}{\partial x_k} ,$$

and g^0_{ik} is its equilibrium value. The difference $g_{ik} - g^0_{ik}$ is the deformation of the lattice of vortices compare with the standard theory of elasticity[6]).

In case of a small deformation

$$g_{ik} - g^{\circ}_{ik} \approx -\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

the expansion of the energy with respect to the displacements has the form:

$$\epsilon = \frac{1}{2}\rho(\mathbf{v}^s - [\vec{\Omega}, \mathbf{r}])^2 + \epsilon_0(\rho) - \frac{1}{2}\rho[\vec{\Omega}, \mathbf{r}]^2$$

$$+ \frac{1}{2}\rho K_1\left(\left(\frac{\partial \mathbf{u}}{\partial x}\right)^2 + \left(\frac{\partial \mathbf{u}}{\partial y}\right)^2\right) + \frac{1}{2}\rho K_2(\vec{\nabla}\mathbf{u})^2 + \frac{1}{2}\rho K_3\left(\frac{\partial \mathbf{u}}{\partial z}\right)^2 . \tag{4}$$

The moduli of elasticity K_1, K_2, and K_3, which are of the order of magnitude of $\kappa\Omega$, must be determined from the microscopic analysis.

The Poisson brackets for the variables $\mathbf{v}^s$ and X_μ can be obtained by examining the Poisson brackets for the coordinates of an isolated vortex, which were determined in Ref. 7. If we introduce the variable X_3 along the vortex line, then these Poisson brackets for the coordinates of the Nth vortex $x_i(X_3,N)$ have the form

$$\{x_i(X_3, N), x_j(X'_3, N')\} = -\frac{1}{\rho\kappa} e_{ijk} \frac{\partial x_k}{\partial X_3} \left|\frac{\partial \mathbf{x}}{\partial X_3}\right|^{-2} \delta_{NN'} \delta(X_3 - X'_3) . \tag{5}$$

A transition to the continual limit is a transition from a discrete variable N associated with a vortex to continuous Lagrangian variables X_1 and X_2, which are rigidly connected with the vortex line in the system of continuously distributed vortices. In this case the Kronecker symbol $\delta_{NN'}$ is converted with a certain coefficient to $\delta(X_1 - X'_1)\delta(X_2 - X'_2)$. Thus, we can go over from a Lagrangian description to an Eulerian description, i.e., from $x_i(X_1,X_2,X_3)$ to $X_\mu(\mathbf{r})$. As a result, we obtain the following universal (i.e., independent of the Hamitonian) Poisson bracket:

$$\{X_1(\mathbf{r}),\, X_2(\mathbf{r}')\} = -\frac{([\vec{\nabla} X_1,\, \vec{\nabla} X_2]\,\mathrm{rot}\,\mathbf{v}^s)}{\rho(\mathrm{rot}\,\mathbf{v}^s)^2}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{6}$$

The other universal Poisson brackets can be obtained analogously:

$$\{X_\mu(\mathbf{r}),\, \mathbf{v}^s(\mathbf{r}')\} = \frac{1}{\rho}\,\vec{\nabla} X_\mu \delta(\mathbf{r}-\mathbf{r}'), \tag{7}$$

$$\{v_i^s(\mathbf{r}),\, v_k^s(\mathbf{r}')\} = -e_{ikl}\,\frac{(\mathrm{rot}\,\mathbf{v}^s)_l}{\rho}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{8}$$

Moreover, we have another nonzero Poisson bracket, which follows from the fact that ρ/m is the density of the gradient transformation operator under whose action v^s is transformed according to the law $\mathbf{v}^s \to \mathbf{v}^s + \hbar/m\nabla\phi$:

$$\{\rho(\mathbf{r}),\, \mathbf{v}^s(\mathbf{r}')\} = \vec{\nabla}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{9}$$

We can see that Eqs. (6)–(9) satisfy the Jacobian identities when condition (1) is satisfied. The total set of equations for the hydrodynamics of rotating He II follows from Eqs. (6)–(9) and the Hamiltonian (3). To calculate the natural modes of rotating He II, we write these equations in linearized form and assume that the displacements $\mathbf{u}$ and the velocity of the rotating coordinate system $\tilde{\mathbf{v}}^s = \mathbf{v}^s - [\mathbf{\Omega},r]$ are small:

$$\dot{\rho} = \{H,\, \rho\} = -\rho\vec{\nabla}\tilde{\mathbf{v}}^s, \tag{10}$$

$$\dot{\tilde{\mathbf{v}}}^s = \{H,\, \tilde{\mathbf{v}}^s\} = -2[\vec{\Omega},\, \tilde{\mathbf{v}}^s] + K_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathbf{u} + K_3\frac{\partial^2}{\partial z^2}u + K_2\vec{\nabla}_\perp(\vec{\nabla}\mathbf{u}) - \vec{\nabla}\frac{\partial\epsilon_0}{\partial\rho}, \tag{11}$$

$$\dot{\mathbf{u}} = \{H,\, \mathbf{u}\} = \tilde{\mathbf{v}}^s - \hat{\mathbf{z}}(\hat{\mathbf{z}}\tilde{\mathbf{v}}^s) + \frac{1}{2\Omega}\left[\hat{\mathbf{z}},\, K_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathbf{u} + K_3\frac{\partial^2}{\partial z^2}\mathbf{u} + K_2\nabla_\perp(\vec{\nabla}\mathbf{u})\right], \tag{12}$$

Solving these equations, we find two modes with the spectrum:

$$\omega_{1,2}^2 = \frac{1}{2}(c^2q^2 + 4\Omega^2) \pm \left(\frac{1}{4}(c^2q^2 + 4\Omega^2)^2 - 4\Omega^2c^2q_z^2 - K_1c^2q_\perp^4\right)^{1/2}. \quad (13)$$

Here c is the velocity of sound

$$\left(c^2 = \rho\frac{\partial^2\epsilon_0}{\partial\rho^2}\right).$$

Expression (13) is valid if the wavelength exceeds the average distance between the vortices $\sim\sqrt{\kappa/\Omega}$, which must be larger than the size of the vortex core $\sim\kappa/c$, i.e.,

$$q << \sqrt{\frac{\Omega}{\kappa}} << \frac{c}{\kappa}. \quad (14)$$

Because of these conditions, the terms containing K_2 and K_3 give small spectrum corrections and hence can be dropped. At $K_1 = 0$ Eq. (13) gives the standard modes in the classical rotating liquid: acoustic wave (at $q \gg \Omega/c$ $\omega_1 = cq$) and inertial mode (at $q \gg \Omega/c\omega_2 = 2\Omega\,|q_z|/q$). The term with K_1 is large only in the second mode when q_z are small. At $q_z = 0$ and $q \gg \Omega/c$ we have a linear spectrum $\omega_2 = K_1^{1/2}q$ of the Tkachenko waves—transverse lattice vibrations.

In conclusion, we note that relations (6)–(8) are difficult to obtain by using a purely phenomenological approach (see Refs. 5 and 8) without investigating the dynamics of an isolated defect. Thus, in Ref. 5 the right-hand side of Eq. (8) is missing; as a result, the equations for the vortex motion of He II in Ref. 5 can be used within the limits of a strong interaction of the vortices with a normal component. Instead of X_μ, a variable, which is canonically conjugated with $\mathbf{v}^s$, was introduced in Ref. 8; however, the equations, which describe the lattice dynamics, were not obtained, because the Poisson brackets for the components of the introduced variable are not known.

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