Soliton quantization

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(Submitted January 6, 1975)
ZhETF Pis. Red. **21**, No. 5, 302–305 (March 5, 1975)

The quasiclassical quantization of particlelike solutions are discussed with the "sine-Gordon" equation as an example, and the quantum corrections are calculated with the aid of a continual integration.

PACS numbers: 03.70.

1. *Quasiclassical results*. We consider the Lagrangian for the chiral field $\chi(x,t)=\exp[iu(x,t)]$

$$L=-\frac{1}{2\gamma}\int_{-\infty}^{\infty}[u_t^2-u_x^2-2m^2(1-\cos u)]\,dx;$$

$$\chi(x,t)\to 1,\quad |x|\to\infty,$$

where m is the mass of the field, and γ is the dimensionless coupling constant ($h=c=1$). In[1,2] is described a nonlinear canonical transformation from the field variables $\{u(x,t);\ \pi(x,t)=(1/\gamma)u_t(x,t)\}$ to the variables of the action—angle type, which combine into the following canonically conjugate pairs: $\{\rho(k);\ \phi(k)\}$, $-\infty<k<\infty$, $0\leqslant\rho(k)<\infty$, $0\leqslant\phi(k)\leqslant 2\pi$, $\{p_a;q_a\}$, $a=1,\ldots,A$, $-\infty\leqslant p_a;\ q_a\leqslant\infty$; $\{\xi_b;\eta_b\}$, $\{\alpha_b;\beta_b\}$; $b=1,\ldots,B$, $-\infty<\xi_b;\eta_b<\infty$; $0\leqslant\alpha_b\leqslant 2\pi$; $0\leqslant\beta_b\leqslant 8\pi/\gamma$, where the positive integers A and B can assume arbitrary values. The total energy and the momentum are expressed in terms of the foregoing variables in the following manner:

$$P_0=\int_{-\infty}^{\infty}(k^2+m^2)^{1/2}\rho(k)\,dk+\sum_{a=1}^{A}(p_a^2+M^2)^{1/2}+\sum_{b=1}^{B}(\eta_b^2+(2M\sin\nu)^2)^{1/2};$$

$$P_1=\int_{-\infty}^{\infty}k\rho(k)\,dk+\sum_{a=1}^{A}p_a+\sum_{b=1}^{B}\eta_b;$$

$$\nu=\gamma\beta/16;\quad M=8m/\gamma.$$

When quantizing in the quasiclassical approximation we can use arbitrary canonical variables, and in particular the ones just described. In this approximation, we can state that the system consists of three sorts of particles with masses m, M, and $2M\sin\nu$. The canonical variables α and β run through the compact phase space. Experience with the theory of the Lie group representations[3] shows that these representations can be quantized only if its total area (number of states), equal in our case to $16\pi^2/\gamma$, is a multiple of 2π. In other words, the coupling constant is quantized $\gamma=8\pi/N$, with N an integer, the eigenvalues ν_n take the form $\pi(n+1/2)/2N$, and the masses of the particles of the third sort are equal to $2M\sin[\pi(n+1/2)/2N]$. This paradoxical result, already noted in[1], is confirmed by an investigation of the quasiclassical expression for the amplitudes of the scattering of the particles of the second sort-solitons. A soliton corresponds to the solution of the equation of motion at $\rho=0$, $A=1$, $B=0$, $u_1(x,t/p,q)=4\tan^{-1}\times(\exp\{\epsilon m(x-vt-q)/\sqrt{1-v^2}\})$, $v=p(p^2+M^2)^{-1/2}$, and ϵ is the charge of the soliton (see[1,2]). The equation of motion has a two-soliton solution $u_2(x,t)$ that breaks up as $t\to\pm\infty$ into a sum of single-soliton solutions with characteristics[4]

$$q_{1+}=q_{1-}+\frac{M}{(p_1^2+M^2)^{1/2}}\ln\left(1-\frac{4M^2}{s}\right);$$

$$q_{2+}=q_{2-}-\frac{M}{(p_2^2+M^2)^{1/2}}\ln\left(1-\frac{4M^2}{s}\right);$$

$$p_{1+}=p_{1-};\quad p_{2+}=p_{2-};\qquad s=(p_{10}+p_{20})^2-(p_1+p_2)^2.$$

These formulas constitute a canonical transformation from the *in* to the *out* variables, i.e., they specify the classical S matrix. The corresponding generating function is

$$p_1q_1 + p_2q_2 - H(p_1; p_2),$$

where

$$H(p_1; p_2) = \frac{16}{\gamma}\int_1^{\xi}\frac{dx}{x}\ln\frac{x+1}{x-1} = i\frac{8}{j}\int_0^{\pi} d\theta \ln\frac{\xi e^{-i\theta}+1}{\xi + e^{-i\theta}} \quad ;$$

$$\xi = \frac{s - 2M^2 + \sqrt{s(s-4M^2)}}{2M^2} .$$

The quantum S matrix in the quasiclassical approximation is given by $\langle p_1; p_2 | S | p_1'; p_2' \rangle = \delta(p_1 - p_1')\delta(p_2 - p_2') \times \exp[-i\epsilon_1\epsilon_2 H]$ and we obtain, depending on the particle charges, two S functions, $S_\pm = \exp[\mp iH(p_1; p_2)]$. The crossing-invariance condition $S_\mp(4M^2 - S) = (-1)^c S_\pm(s)$ with integer c is satisfied only if γ is quantized in the manner indicated above. The expression for $S_\pm(s)$ in not real in the unphysical region $0 \le s \le 4M^2$, in contradiction to the general principles of analyticity. We hope that the quantum corrections will improve the situation. Indeed, by approximating the integral with integral sums and by assuming γ to be quantized, we obtain for S_+ the expression

$$S_+(s) = \exp\left\{\frac{N}{\pi}\int_0^{\pi} d\theta \ln\frac{\xi e^{-i\theta}+1}{\xi + e^{-i\theta}}\right\} = \prod_{n=0}^{N-1}\frac{\xi e^{-i\theta_n}+1}{\xi + e^{-i\theta_n}} \; ;$$

$$\theta_n = \frac{\pi}{N}\left(n + \frac{1}{2}\right) ,$$

which has the correct analytic properties in the case when the pole is attracted to the points $S = (2m \sin\nu_n)^2$—in other words, the particles of the third sort (double solitons) are bound states of solitons.

2. *Quantum corrections*. To investigate them it is convenient to use a continual integral. Let us illustrate this using as an example the corrections to the soliton mass. The Green's function $G(p_1; p_2; t_1 - t_2)$, which describes the transition from the state "soliton with momentum p_1 at $t = t_1$" into the state "soliton with momentum p_2 at $t = t_2$," is given as $T = t_1 - t_2 \to \infty$ by the integral

$$\int \prod_{x,t} du\, d\pi \exp\{ i\int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} dx(\pi u_t - H[u;\pi])\};$$

$$H[u;\pi] = \frac{\gamma}{2}\pi^2 + \frac{1}{2\gamma}u_x^2 + \frac{1-\cos u}{\gamma} ,$$

where $u = u_1(x, t | p_i q_i)$ at $t = t_i$, with the limit independent of q_1 and q_2. To calculate the coefficient F at $\delta(p_1 - p_2)$ it is convenient to integrate over a submanifold with fixed total momentum. The natural additional condition associated with the constraint $\int \pi u_x dx + p = 0$ takes the form $X \equiv \int dx\, xH / \int dxH = f(t)$. The general prescription[5] shows that the sought coefficient $F[(p^2 + M^2)^{1/2}MT]$ is given by the integral

$$\int \exp\{ i\int_{t_1}^{t_2} dt \int dx(\pi u_t - H[u;\pi])\} \prod_t \delta(P_1[u;\pi] - p)\delta(X[u;\pi] - f(t)) \times \prod_{x,t} d\pi\, du.$$

It suffices to calculate it at $p = f(t) = 0$. We assume $u = u_1(x, t\cdot 10.0) + \gamma^{1/2}w(x,t)$ and $\pi = \gamma^{-1/2}v(x,t)$, and confine ourselves at first to the orders γ^{-1} and γ^0 (trees and one loop). The answer is

$$F(T) = \exp\{-iMT\}\exp - \frac{1}{2}\,\mathrm{Tr}'\ln\left(\frac{d^2}{dt^2} + D\right) = F_{-1}F_0 ,$$

where $D = -(d^2/dx^2) + m^2 - 2m^2/\cosh^2 mx$. The symbol Tr' means that when the trace is taken, one leaves out the contribution of the eigenfunction $\psi_0(x)$ of the operator D with zero eigenvalue. This limitation is brought about by the condition with the additional condition. As $T \to \infty$, using the known formulas for the traces[6] of the Schrödinger operator, we have

$$F_0 = \exp\{-i\frac{1}{2}\mathrm{tr}' D^{1/2}T\} = \exp\left\{\frac{iM}{4}\frac{\gamma}{(2\pi)^2}\int\frac{d^2k}{k^2 + m^2}T\right\},$$

where the trace tr' is taken only in x space. The resultant divergence is eliminated by the same renormalization of m which eliminates the divergence in ordinary perturbation theory in this approximation.[7] (The equality of these renormalizations was pointed out to us by A.M. Polyakov.) We see that the quantum correction in the single-loop approximation has been reduced only to a renormalization. The multiloop corrections may make a nontrivial contribution to the soliton mass. The calculation of the latter becomes simplified if one uses t'Hooft's stratagem, replacing $\delta(X - f(t))$ in the continual integral by $\exp[-i\mu^2 M \int dtX^2]$. Perturbation theory is now constructed with a propagator $\Delta(x_1t_1/x_2t_2)$ such that $(d^2 dt_1^2 + D + \mu^2\mathcal{P})\Delta = \delta(x_1 - x_2)\delta(t_1 - t_2)$, where $\mathcal{P}$ is the projector on $\psi_0(x)$, and $\Delta(x_1t_1/x_2t_2)$ does not contain the infrared divergences that hinder other other approaches to soliton quantization.[8]

The foregoing calculations show that the quantum corrections are identical with respect to the quantum constant and that the quasiclassical answer is a good approximation at small γ.

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